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INVITED TALKS

FINDING A SUBDIVISION OF A DIGRAPH

Frédéric Havet

MASCOTTE Project, CNRS-INRIA Sophia-Antipolis, France

We consider the following problem: Given a directed graph D , does it contain a subdivision of a prescribed digraph F ? We give a number of examples of polynomial instances, several NP-completeness proofs as well as a number of conjectures and open problems.

This is joint work with J. Bang-Jensen and A.-K. Maia.

EXTENDING PARTIAL REPRESENTATIONS OF GRAPHS

Jan Kratochvíl

Charles University, Prague, Czech Republic

We study the computational complexity of the following general question: Given a graph class defined by geometric representations of a certain type, and given a graph and a representation of some of its vertices, can this be extended to a representation of the entire graph? This falls into a general paradigm of extending a partial solution of a problem to a full one which was studied in various connections (coloring, noncrossing drawing etc.). In many cases the extension problem is provably harder which corresponds to a common knowledge of architects - building from scratch is easier than restoring an old house. The results we will report upon include extension of geometrically defined classes of intersection graphs - interval graphs, permutation graphs, function graphs, and chordal graphs. Perhaps slightly suprisingly we will see that in most cases the extension problem remains polynomial.

PLAYING FOR TIME: 1101110 YEARS OF COMBINATORIAL GAME THEORY

Richard J. Nowakowski

Dalhousie University, Halifax, Canada

Combinatorial Game Theory (CGT) is a new mathematical subject despite being a very old human endeavour. I'll give a brief, admittedly biased, overview of the ideas (such as Hot games, incentives, atomic weight, flowers, miserable monoids), problems, and people, from Bouton's 1902 paper on NIM to the first comprehensive CGT text (soon-to-be-published by Aaron Siegel). The talk will assume no specialist knowledge of CGT.

THE CYCLE SPECTRUM OF GRAPHS

Dieter Rautenbach

Universität Ulm, Ulm, Germany

Cycles in graphs and especially their lengths are among the most well-studied topics in graph theory. The set of all cycle lengths of a graph is known as its cycle spectrum. We present recent results concerning

- the cycle spectrum of Hamiltonian graphs,
- the cycle spectrum of squares of graphs,
- a short proof of a versatile version of Fleischner's well-known theorem, and Ramsey-type results for the cycle spectrum.

The talk is based on joint work with S. Brandt, F. Joos, D. Meierling, K. Milans, J. Müttel, F. Regen, T. Sasse, D. West.

GRAPHS, TOURNAMENTS, COLOURING AND CONTAINMENT

Paul Seymour

Princeton University, Princeton, USA

Some tournaments H are “heroes”; they have the property that all tournaments not containing H as a subtournament have bounded chromatic number (colouring a tournament means partitioning its vertex-set into transitive subsets). In joint work with eight authors, we found all heroes explicitly. That was great fun, and it would be nice to find an analogue for graphs instead of tournaments. The problem is too trivial for graphs, if we only exclude one graph H ; but it becomes fun again if we exclude a finite set of graphs. The Gyrfas-Sumner conjecture says that if we exclude a forest and a clique then chromatic number is bounded. So what other combinations of excluded subgraphs will give bounded chromatic (or cochromatic) number? It turns out (assuming the Gyrfas-Sumner conjecture) that for any finite set S of graphs, the graphs not containing any member of S all have bounded cochromatic number if and only if S contains a complete multipartite graph, the complement of a complete multipartite graph, a forest, and the complement of a forest. Proving this led us to the following: for every complete multipartite graph H , and every disjoint union of cliques J , there is a number n with the following property. For every graph G , if G contains neither of H, J as an induced subgraph, then $V(G)$ can be partitioned into two sets such that the first contains no n -vertex clique and the second no n -vertex stable set. In turn, this led us (with Alex Scott) to the following stronger result. Let H be the disjoint union of H_1, H_2 , and let J be obtained from the disjoint union of J_1, J_2 by making every vertex of J_1 adjacent to every vertex of J_2 . Then there is a number n such that for every graph G containing neither of H, J as an induced subgraph, $V(G)$ can be partitioned into n sets such that for each of them, say X , one of H_1, H_2, J_1, J_2 is not contained in $G|X$. How about a tournament analogue of this? It exists, and the same (short) proof works; and this leads to a short proof of the most difficult result of the heroes paper that we started with. There are a number of other related results and open questions. Joint work with Maria Chudnovsky.

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On average fault-tolerance in product graphs

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EXTENDED ABSTRACT

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The cardinalities of these sets are denoted by $|V(G)| = n$ and $|E(G)| = e$. Let u and v be two distinct vertices of G . A path from u to v , also called an uv -path in G , is a subgraph P with vertex set $V(P) = \{u = x_0, x_1, \dots, x_r = v\}$ and it is usually denoted by $P : x_0x_1 \cdots x_r$. Two uv -paths P and Q are said to be internally disjoint if $V(P) \cap V(Q) = \{u, v\}$. A cycle in G is a path $C : x_0x_1 \cdots x_r$ such that $x_0 = x_r$. The *girth* of G , denoted by $g(G)$, is the length of a shortest cycle in G , and if G contains no cycles, then $g(G) = \infty$. The set of adjacent vertices to $v \in V(G)$ is denoted by $N_G(v)$. The *degree* of v is $d_G(v) = |N_G(v)|$, whereas $\delta(G) = \min_{v \in V(G)} d_G(v)$ and $\bar{d}(G) = \frac{1}{n} \sum_{v \in V(G)} d_G(v) = 2e/n$ are the *minimum degree* and the *average degree* of G , respectively. The connectivity $\kappa(G)$ of a graph G is the smallest number of vertices whose deletion from G produces a disconnected or a trivial graph. Whitney [10] proved in 1932 that a graph G is r -connected, that is, $\kappa(G) \geq r$, if and only if every pair of vertices is connected by r internally disjoint paths. From this result, we know that the connectivity $\kappa_G(u, v)$ between two distinct vertices u and v in G is the maximum number of pairwise internally disjoint uv -paths in G . In this way, the connectivity of a graph can be seen as $\kappa(G) = \min_{u, v \in V(G)} \kappa_G(u, v)$. In [10] the author also showed that $\kappa(G) \leq \delta(G)$. The graph G is *maximally connected* if the previous bound is attained, that is, if $\kappa(G) = \delta(G)$.

For a graph G of order n , the *average connectivity* $\bar{\kappa}(G)$ is defined as the average of the connectivities between all pairs of vertices of G , that is,

$$\bar{\kappa}(G) = \frac{1}{\binom{n}{2}} \sum_{u, v \in V(G)} \kappa_G(u, v).$$

In order to avoid fractions, we also consider the *total connectivity* $K(G)$ of G , defined as $K(G) = \sum_{u, v \in V(G)} \kappa_G(u, v)$. While the connectivity is the minimum number of vertices

whose removal separates at least one connected pair of vertices, the average connectivity is a measure for the expected number of vertices that have to be removed to separate a randomly chosen pair of vertices.

It is well known that most networks can be modeled by a graph $G = (V, E)$. The best known measure of reliability of a graph is its connectivity, defined above. As the connectivity is a worst-case measure, it does not always reflect what happens throughout the graph. For example, a tree and the graph obtained by appending an end-vertex to a complete graph both have connectivity 1. Nevertheless, for large order the latter graph is far more reliable than the former. Interest in the vulnerability and reliability of networks such as transportation and communication networks, has given rise to a host of other measures of reliability, see for example [1]. In this paper we pay attention to a measure for the reliability of a graph, the *average connectivity*, introduced by Beineke, Oellermann and Pippert [3].

There is a lot of research on the connectivity of a graph (see [8]). Many works provide sufficient conditions for a graph to be maximally connected or super connected [4]. Others study the maximal connectivity in networks that are constructed from graph generators, as cartesian product graphs [5, 9] or permutation graphs [2, 7].

There are two excellent papers where the average connectivity has been investigated. In the first one, Beineke, Oellermann and Pippert [3] find upper and lower bounds on the average connectivity of a graph G in terms of its order n and its average degree $\bar{d}(G)$. In the second one, Dankelmann and Oellermann [6] obtain sharp upper bounds for some families of graphs. We study the average connectivity of the so-called *strong product of graphs*.

For a large system, configuration processing is one of the most tedious and time-consuming parts of the analysis. Different methods have been proposed for configuration processing and data generation. Some of them are structural models which can be seen as the product graph of two given graphs, known as generators. Many properties of structural models can be obtained by considering the properties of their generators. In this sense, a usual objective in network design is the extension of a given interconnection system to a larger and fault-tolerant one so that the communication delay among nodes of the new network is small enough. To achieve this goal, many works in Graph Theory have studied fault-tolerant properties of some products of graphs.

We focus on the *strong product* $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 is defined on the cartesian product of the vertex sets of the generators, so that two distinct vertices (x_1, x_2) and (y_1, y_2) of $G_1 \boxtimes G_2$ are adjacent if $x_1 = y_1$ and $x_2 y_2 \in E(G_2)$, or $x_1 y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_1 y_1 \in E(G_1)$ and $x_2 y_2 \in E(G_2)$.

In this work we provide, by a constructive method, a lower bound on the average connectivity of the strong product $G_1 \boxtimes G_2$ of two connected graphs G_1 and G_2 of girth at least 5. As a consequence, we prove that the strong product of two maximally connected graphs of girth at least 5 is maximally connected, and also, that $\bar{\kappa}(G_1 \boxtimes G_2) = \bar{d}(G_1 \boxtimes G_2)$ if $\bar{\kappa}(G_i) = \bar{d}(G_i)$, $i = 1, 2$.

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Excluding 4-wheels

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EXTENDED ABSTRACT

A k -wheel is a graph formed by a cycle C , called the *rim*, and a vertex u (not in $V(C)$), called the *center*, such that u has at least k neighbors in C .

We say that a graph G contains a graph H if H is isomorphic to a subgraph of G . If a graph G does not contain a graph H we say that G is H -free. If \mathcal{H} is a class of graphs we say that a graph G is \mathcal{H} -free if for any $H \in \mathcal{H}$, G is H -free. We say that two non-adjacent vertices u and v are *twins* if $N(u) = N(v)$.

In [4] Thomassen and Toft proved the following result (an alternative proof is given in [1]).

Theorem 1 *If G is a 3-wheel-free graph, then either it contains a pair of twins or it contains a vertex of degree at most 2.*

From this Theorem they easily get the following corollary that settles a conjecture proposed by Toft in [5].

Corollary 2 *If G is a 3-wheel-free graph, then G is 3-colorable.*

In [6] Turner proved the following general result on k -wheel-free graphs for any $k \geq 4$.

Theorem 3 *For any integer $k \geq 4$, if G is a k -wheel-free graph, then G contains a vertex of degree at most k .*

Which has the following corollary :

Corollary 4 *For any integer $k \geq 4$, if G is a k -wheel-free graph, then G is $(k+1)$ -colorable.*

In this paper we extend Theorem 1 and strengthen Theorem 3 for $k = 4$ by proving the following result.

Theorem 5 *If G is a 4-wheel-free graph, then either it contains a pair of twins or it contains a vertex of degree at most 3.*

Which implies the following :

Corollary 6 *If G is a 4-wheel-free graph, then G is 4-colorable.*

We conjecture the following :

Conjecture 7 *If G is a k -wheel-free graph, then it is k -colorable.*

Regarding this subject, one can also find some extremal result on 3-wheel-free graphs in [3] and on 4-wheel-free graphs in [2]. In [1], the class of graphs that does not contain 2-wheels as induced subgraphs is studied. The authors give a structure theorem for this class using the class of 2-wheel-free graph as a basic class.

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Graphs with no cycle with a fixed number of chords

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EXTENDED ABSTRACT

Given a graph G , a vertex coloring of the graph is called *proper* if for every pair of adjacent vertices x, y the color assigned to x is different from the color assigned to y . The *chromatic number* of G denoted by $\chi(G)$ is the minimum number of colors needed to properly color the graph. Computing the chromatic number of a graph is NP-complete. Even to decide if a planar graph is 3-colorable is NP-complete. In addition, it is difficult to approximate in polynomial time the chromatic number of a graph up to a factor $n^{1-\epsilon}$ unless $P = NP$. The maximum size of a clique of G , denoted by $\omega(G)$, is a lower bound on $\chi(G)$. Indeed, the vertices of a clique must have distinct colors. Does it exist a function f linking the chromatic number of a graph with its maximal clique? Erdős proved that there are some graphs with arbitrarily large chromatic number and arbitrarily large girth [1], i.e. such that $\omega(G) = 2$. Hence such a function f does not exist for every graph. In the following we will only consider classes closed under induced subgraph. A class of graphs \mathcal{C} is said to be χ -*bounded* if there exists a function f such that for every graph $G \in \mathcal{C}$, $\chi(G) \leq f(\omega(G))$. The notion of χ -boundedness was introduced by Gyárfás in [2] and lots of research has been done in this domain (see [4] for a survey).

It is interesting to determine which class of graphs has to be forbidden in order to ensure χ -boundedness.

First consider the case when a unique graph H is forbidden as an induced subgraph. For instance, Gyárfás proved that for every k , the class of graphs with no induced k -stars is χ -bounded, as the class of graphs with no induced paths of length k . Since there exist graphs with arbitrarily large girth and chromatic number, if the graph H contains a cycle, the class is not χ -bounded. And Gyárfás conjectured in [2] that it is a sufficient condition, i.e. that the class of graphs with no induced subgraph H is χ -bounded if and only if H is a forest.

Secondly, we can forbid the induced subdivision of a graph H . The graph H is said to be forbidden as an *induced topological minor*. For instance, a graph with no induced subdivision of a triangle has a bounded chromatic number. Indeed such a graph is cycle-free, i.e. is a forest.

In this paper we consider a third type of forbidden structures which combines the two first ones. The forbidden structure is a graph H for which some edges can be subdivided but some cannot. Such a forbidden structure is sometimes called an *induced semi-topological minor*. To our knowledge, the only known χ -boundedness result about a class of graph defined by forbidding an induced semi-topological minor is the following. In [3], Trotignon and Vušković proved that the class of graphs with no induced cycle with a unique chord satisfies $\chi(G) = \max(\omega(G), 3)$. Forbidding cycles with a unique chord is equivalent to forbid a diamond (a diamond is a C_4 with a diagonal) such that every edge but the diagonal can be subdivided.

A k -*cycle* is an induced cycle with exactly k chords. Our results extend the results of [3] since we study the class of graphs with no 2-cycle (Theorem 1), and with no 3-cycle (Theorem 2).

Theorem 1 *A graph with no 2-cycle has chromatic number at most 6.*

Actually we prove a slightly stronger result. Indeed, there are types of 2-cycles detailed on Fig. 1. Nevertheless, we prove that the class of graphs with no cycle of type (a) and no cycle of type (b) has chromatic number at most 6. Note that the maximum clique number

for graphs with no 2-cycle is at most 3. Indeed, if there is a K_4 , the graph of Fig. 1(b) appears.

The proof of this result is a mix of two methods. First we prove that some structures can be avoided with some decomposition steps. For instance the graph can be considered as diamond-free. It simplifies the structure of the graph we study.

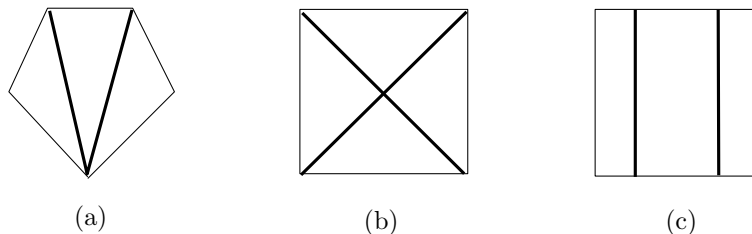


Figure 1: The three types of 2-cycle. Thin edges can be subdivided but not bigger ones.

The second step is somehow an induction step on the number of chords. Given a vertex x , the k -th level of x is the set of vertices at distance exactly k from x . If a connected graph G has chromatic number χ and x is a vertex of the graph, one of these levels has chromatic number at least $\lceil \chi/2 \rceil$. Therefore if $\chi(G) \geq 7$, then [3] ensures that there exists an integer k such that the k -th neighborhood contains a 1-cycle. This 1-cycle can be transformed into a 2-cycle using the level partition.

Then we also extend this result to graphs with no 3-cycle

Theorem 2 *There exists a constant c such that every graph G with no 3-cycle satisfies:*

$$\chi(G) \leq c \cdot \omega(G).$$

The proof of Theorem 2 follows the same scheme as proof of Theorem 1 except this it is more technical. The results of Trotignon and Vušković [3] and our results lead to this more general question.

Conjecture 3 *Let k be an integer. Is the class of graphs with no k -cycle χ -bounded?*

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Clock-free graphs and k -clock-free graphs of high girth

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EXTENDED ABSTRACT

A *clock* is a graph consisting of a (chordless) cycle C and a vertex with exactly two neighbours on C (see Figure 2). We study the class of *clock-free* graphs, graphs without induced clocks, and consider a conjecture of Trotignon.

Conjecture 1 [5] *Every clock-free graph which contains no K_3 and no cube has a vertex of degree at most 2.*

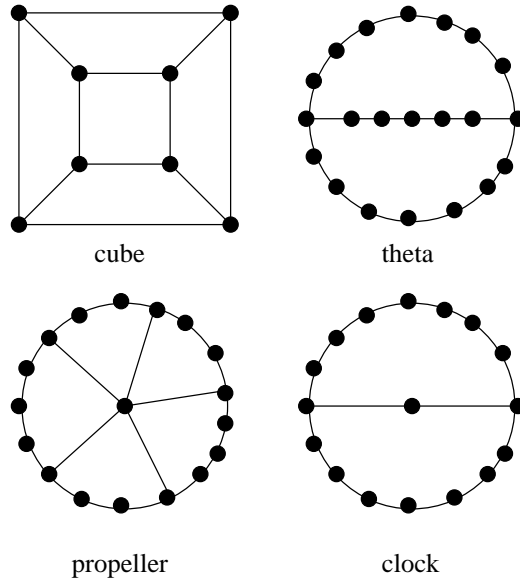


Figure 2: A cube, a theta, a propeller and a clock

We answer this conjecture in the positive for graphs of high girth. Note that graphs with girth at least 5 contain no cube.

Theorem 2 *Every 2-connected clock-free graph of girth at least 10 has a vertex of degree at most 2.*

This conjecture generalizes recent theorems for two classes of graphs.

This first theorem arises from the study of minimally k -connected graphs (a k -connected graph where the removal of any edge disconnects the graph). The class of chordless graphs contain all minimally 2-connected graphs and their subgraphs [2, 3]. Graphs in this class always contain at least two vertices of degree 2 (when the graph has at least two vertices) [2, 3]. Aboulker, Radovanović, Trotignon and Vušković[1] showed an analogous result for vertex connectivity. They show propeller-free graphs (graphs that do not contain an induced cycle C and a vertex with at least two neighbours on C as a subgraph) also contain a degree 2 vertex.

Theorem 3 [1] *Every propeller-free graph contains a vertex of degree at most 2.*

The second theorem Conjecture 1 generalizes concerns graphs excluding an induced *theta*, a subdivision of $K_{2,3}$.

Theorem 4 [4] *Every graph which contains no induced theta, no K_3 and no cube has a vertex of degree at most 2.*

Note that a clock is both a propeller and a theta (in triangle free graphs) and that a cube contains a propeller. Thus, Conjecture 1 generalizes both Theorem 3 and 4.

We further extend our result to graphs without k -clocks, an induced subgraph consisting of a cycle C and a path with k edges between two vertices of C .

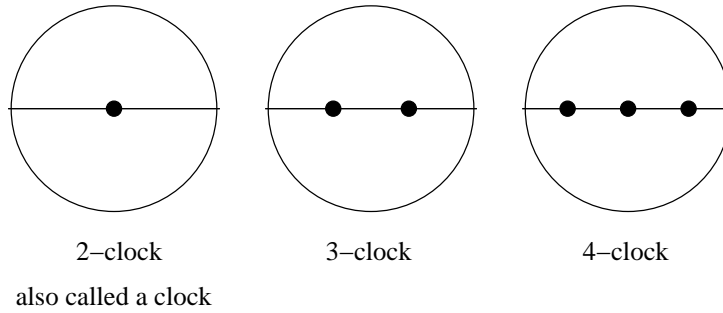


Figure 3: k -clocks

Theorem 5 *For every $k \geq 2$, every k -clock-free graphs of girth at least $6k$ has a vertex of degree at most 2.*

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Linear balanceable and subcubic balanceable graphs

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EXTENDED ABSTRACT

A bipartite graph G is *balanced* if and only if every hole of G has length $0 \pmod{4}$, where a *hole* is a chordless cycle of length at least 4. These graphs came from *balanced matrices*. Balanced matrices have been studied extensively in literature, due to their important polyhedral properties, for a survey see [4]. A *signed bipartite graph* is a bipartite graph, together with an assignment of weights $+1, -1$ to the edges of G . A signed bipartite graph is *balanced* if the *weight* of every hole H of G , i.e. the sum of the weights of the edges of H , is $0 \pmod{4}$. A bipartite graph is *balanceable* if there exists a *signing* of its edges, i.e. an assignment of weights $+1, -1$ to the edges of the graphs, such that the resulting signed bipartite graph is balanced.

The following conjecture is the last unresolved conjecture about balanced (balanceable) bipartite graphs in Cornuéjols' book [7] (it is Conjecture 6.11). Note that Conjectures 9.23, 9.28 and 9.29 from [7] have been resolved by Chudnovsky and Seymour in [2].

Conjecture 1 (Conforti and Rao [6]) *Every balanced bipartite graph contains an edge that is not the unique chord of a cycle.*

In other words, every balanced bipartite graph contains an edge whose removal leaves the graph balanced. This is not true if the graph is balanceable, as shown by R_{10} , that is the graph defined by the cycle $x_1x_2 \dots x_{10}x_1$ (of length 10) with chords x_ix_{i+5} , $1 \leq i \leq 5$. Graph R_{10} is cubic and balanceable (a proper signing of R_{10} is to assign weight $+1$ to the edges of the cycle $x_1x_2 \dots x_{10}x_1$ and -1 to the chords), but not balanced ($x_1x_2x_3x_4x_5x_6$ is a hole of length 6). Note that in R_{10} every edge is the unique chord of some cycle. Conjecture 1 generalizes to balanceable graphs in the following way.

Conjecture 2 (Conforti, Cornuéjols and Vušković [4]) *In a balanceable bipartite graph either every edge belongs to some R_{10} or there is an edge that is not the unique chord of a cycle.*

A bipartite graph is *linear balanceable* if it is balanceable and does not contain a 4-hole (i.e. a hole of length 4). A graph G is *subcubic* if $\Delta(G) \leq 3$. We will prove [1] that conjectures 1 and 2 hold when restricted to linear balanceable graphs

Theorem 3 *Let G be a linear balanceable graph that has at least one edge. Then there is an edge of G that is not the unique chord of a cycle.*

For the subcubic case, we proved a result conjectured by Morris, Spiga and Webb [8], stating that every cubic balanced bipartite graph contains a pair of vertices with the same neighborhood.

Theorem 4 *If G is a cubic balanceable graph that is not R_{10} , then G has a pair of twins none of whose neighbors is a cut vertex of G .*

As it was noticed in [8] (for the special case of cubic balanced graphs), Theorem 4 implies the following.

Corollary 5 *Let G be a cubic balanceable graph. Then the following hold:*

- (i) G has girth four.
- (ii) If $G \neq R_{10}$ then G contains an edge that is not the unique chord of a cycle.
- (iii) G is not planar.

This prove the conjectures 1 and 2. (Note that if there is a vertex of degree at most 2 the result is clear).

Our proofs use the following decomposition theorem, obtained mixing results from Conforti, Cornuéjols, Kapoor and Vušković [3], Conforti and Rao [5] and Yannakakis [11].

Theorem 6 *Let G be a connected balanceable graph.*

- *If G is 4-hole-free, then G is basic, or has a 2-join, a 6-join or a star cutset.*
- *If $\Delta(G) \leq 3$, then G is basic or is R_{10} , or has a 2-join, a 6-join or a star cutset.*

We use Theorem 6 in our proof of Conforti and Rao Conjecture in the subcubic case, instead of Seymour's decomposition theorem for regular bipartite graphs [9] because we have only to check whether three cutsets (2-join, 6-join and star cutset) go through our induction hypothesis, whereas if we used the decomposition theorem in [9] we would have to check five cutsets (1-join, 2-join, 6-join, N-join and M-join, for an explanation see [10]). Furthermore, 2-joins and 6-joins in graphs with no star cutset have special properties which are very useful for pushing the induction hypothesis through them.

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Stress freeness of K_d -minor free graphs

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EXTENDED ABSTRACT

The motivation for this paper is a problem that arises from the study of tension and compression forces applied on frameworks in the Euclidian space \mathbb{R}^d . A d -framework is a graph $G = (V, E)$ and an embedding ρ of G in \mathbb{R}^d . The reader should think of a framework as an actual physical system where edges are either straight bars or cables and vertices are articulated joints.

A *stress* on a framework (G, ρ) is a function $\omega : V \times V \rightarrow \mathbb{R}$ such that $\forall u \in V$,

$$\sum_{\{u,v\} \in E} \omega(\{u,v\})(\rho(v) - \rho(u)) = 0.$$

Stress corresponds to some notion of equilibrium for the associated physical system. Each vertex is affected by tension and compression forces created by the bars and cables. $\omega(\{u,v\})$ can be thought of as the magnitude of such force per unit length, with $\omega(\{u,v\}) < 0$ for a cable tension and $\omega(\{u,v\}) > 0$ for a bar compression. A stress is a state of the system where these forces cancel each other at every vertex. We can see that every framework admits a *trivial* stress where ω is identically zero. A d -framework admitting only the trivial stress is called *d-stress free*.

To make this notion independent of the embedding of G , we introduce the following. A graph G is *generically d-stress free* if the set of all d -stress free embeddings of G in \mathbb{R}^d is open and dense in the set of all its embeddings (i.e. every stressed embedding of G is arbitrary close to a stress free embedding).

This notion has been first used on graphs coming from 1-skeletons of 3-dimensional polytopes [3, 4, 5, 9], which are planar by Steiniz's theorem. Gluck generalized the results on 3-dimensional polytopes to the whole class of planar graphs.

Theorem 1 (Gluck, 1975, [2]) *Planar graph are generically 3-stress free.*

Nevo proved that we can generalize Theorem 1 for K_5 -minor free graphs, and extended the result as follows.

Theorem 2 (Nevo, 2007, [1]) *For $2 \leq r \leq 6$, every K_r -minor free graph is generically $(r-2)$ -stress free.*

He conjectured this to hold also for $r = 7$. We answer positively to Nevo's conjecture.

Theorem 3 *Every K_7 -minor free graph is generically 5-stress free.*

A result of Whiteley [10] was used to derive Theorem 2 from a consideration on graphs whose every edge belongs to at least k triangles, as follows.

Theorem 4 (Nevo, 2007, [1]) *If G has an edge and each edge belongs to at least four triangles, then either G has a K_6 -minor, or G is a clique-sum over K_r , for $r \leq 4$.*

We strengthened Theorem 4 for K_6 -minor free graphs and extend it for K_7 -minor free graphs as such.

Theorem 5 *Any K_6 -minor free (resp. K_7 -minor free) graph G has a vertex u such that $\deg(u) \leq 7$ (resp. $\deg(u) \leq 9$) and a vertex $v \in N(u)$ such that uv belongs to at most three (resp. four) triangles.*

We use it to deduce the following corollary.

Corollary 6 *If G has an edge and each edge belongs to at least five triangles, then G has a K_7 -minor.*

The proof relies heavily on the fact that any K_6 -minor free (resp. K_7 -minor free) contains at most $4n - 10$ (resp. $5n - 15$) edges by Mader's theorem [6]. Moreover any generically d -stress free graph has at most $dn - \binom{d+1}{2}$ edges. Nevo pointed out that Theorem 2 cannot be extended to the case $r = 8$ because of $K_{2,2,2,2,2}$ which is K_8 -minor free but contains too many edges to be generically 6-stress free. Jørgensen [7] proved that any K_8 -minor free graph has at most $6n - 21$ edges or is constructed iteratively by taking disjoint copies of $K_{2,2,2,2,2}$ and identifying cliques of size five. We show the following.

Theorem 7 *Every K_8 and $K_{2,2,2,2,2}$ -minor free graph is generically 6-stress free.*

The proof relies on a similar result to Corollary 6 given by the following theorem.

Theorem 8 *If G has an edge and each edge belongs to at least 6 triangles, then G has a K_8 -minor or a $K_{2,2,2,2,2}$ -minor.*

Song and Thomas [8] proved that the only K_9 -minor free graph with at least $7n - 27$ edges are the graphs constructed iteratively by taking disjoint copies of $K_{1,2,2,2,2,2}$ and identifying cliques of size six, and the graph $K_{2,2,2,3,3}$. This leads us to the following conjecture.

Conjecture 9 *Every K_9 , $K_{2,2,2,2,2,1}$ and $K_{2,2,2,3,3}$ -minor free graph is generically 7-stress free.*

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Generalizing the Strong Perfect Graph Theorem to digraphs

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EXTENDED ABSTRACT

Replacing the chromatic number by the dichromatic number introduced by Neumann-Lara [6] we generalize the notion of perfectness of a graph to digraphs. We give a characterization of perfect digraphs using the notion of perfect graphs. Applying the Strong Perfect Graph Theorem [3], this yields a characterization of perfect digraphs by a set of forbidden induced subdigraphs. Furthermore, modifying a recent proof of Bang-Jensen et al. [1] we show that the recognition of perfect digraphs is $\text{co-}\mathcal{NP}$ -complete.

First we fix some notation. We only consider digraphs without loops. The *clique number* $\omega(D)$ of a digraph D is the size of the largest bidirectionally complete subdigraph of D . The *dichromatic number* $\chi(D)$ of D is the smallest cardinality $|C|$ of a colour set C , so that it is possible to assign a colour from C to each vertex of D such that for every colour $c \in C$ the subdigraph induced by the vertices coloured with c is acyclic, i.e. it does not contain a directed cycle. The clique number is an obvious lower bound for the dichromatic number. D is called *perfect* if, for any induced subdigraph H of D , $\chi(H) = \omega(H)$.

An (undirected) graph $G = (V, E)$ can be considered as the symmetric digraph $D_G = (V, A)$ with $A = \{(v, w), (w, v) \mid vw \in E\}$. In the following, we will not distinguish between G and D_G . In this way, the dichromatic number of a graph G is its chromatic number $\chi(G)$, the clique number of G is its usual clique number $\omega(G)$, and G is perfect as a digraph if and only if G is perfect as a graph. For us, an *edge* vw in a digraph $D = (V, A)$ is the set $\{(v, w), (w, v)\} \subseteq A$ of two antiparallel arcs, and a *single arc* in D is an arc $(v, w) \in A$ with $(w, v) \notin A$. The *oriented part* $O(D)$ of a digraph $D = (V, A)$ is the digraph (V, A_1) where A_1 is the set of all single arcs of D , and the *symmetric part* $S(D)$ of D is the digraph (V, A_2) where A_2 is the union of all edges of D . Obviously, $S(D)$ is a graph, and by definition we have

Lemma 1 *For any digraph D , $\omega(D) = \omega(S(D))$.*

An *odd hole* is an undirected cycle C_n with an odd number $n \geq 5$ of vertices. An *odd antihole* is the complement of an odd hole (without loops). A *filled odd hole/antihole* is a digraph H , so that $S(H)$ is an odd hole/antihole. For $n \geq 3$, the directed cycle on n vertices is denoted by \vec{C}_n . Furthermore, for a digraph $D = (V, A)$ and $V' \subseteq V$, by $D[V']$ we denote the subdigraph of D induced by the vertices of V' .

Theorem 2 *A digraph $D = (V, A)$ is perfect if and only if $S(D)$ is perfect and D does not contain any directed cycle \vec{C}_n with $n \geq 3$ as induced subdigraph.*

Proof. *Proof.* Assume $S(D)$ is not perfect. Then there is an induced subgraph $H = (V', E')$ of $S(D)$ with $\omega(H) < \chi(H)$. Since $S(D[V']) = H$, we conclude by Lemma 1,

$$\omega(D[V']) = \omega(S(D[V'])) = \omega(H) < \chi(H) = \chi(S(D[V'])) \leq \chi(D[V']),$$

therefore D is not perfect. If D contains a directed cycle \vec{C}_n with $n \geq 3$ as induced subdigraph, then D is obviously not perfect, since $\omega(\vec{C}_n) = 1 < 2 = \chi(\vec{C}_n)$.

Now assume that $S(D)$ is perfect but D is not perfect. It suffices to show that D contains an induced directed cycle of length at least 3. Let $H = (V', A')$ be an induced subdigraph of D such that $\omega(H) < \chi(H)$. Then there is a proper colouring of $S(H) = S(D)[V']$ with $\omega(S(H))$ colours, i.e., by Lemma 1, with $\omega(H)$ colours. This cannot be a feasible colouring for the digraph H . Hence there is a (not necessarily induced) monochromatic directed cycle \vec{C}_n with $n \geq 3$ in $O(H)$. Let C be such a cycle of minimal length. C cannot have a chord

that is an edge vw , since both terminal vertices v and w of vw are coloured in distinct colours. By minimality, C does not have a chord that is a single arc. Therefore, C is an induced directed cycle (of length at least 3) in H , and thus in D . \square

Corollary 3 *If D is a perfect digraph, then any feasible colouring of $S(D)$ is also a feasible colouring for D .*

By the Strong Perfect Graph Theorem [3] and Theorem 2 we obtain:

Corollary 4 *A digraph $D = (V, A)$ is perfect if and only if it does neither contain a filled odd hole, nor a filled odd antihole, nor a directed cycle \vec{C}_n with $n \geq 3$ as induced subdigraph.*

Corollary 3 and the fact that k -colouring of perfect graphs is in \mathcal{P} (see [4]) implies the following.

Corollary 5 *k -colouring of perfect digraphs is in \mathcal{P} for any $k \geq 1$.*

To test whether D does not contain an induced directed cycle \vec{C}_n , $n \geq 3$, is a co- \mathcal{NP} -complete problem by a recent result of Bang-Jensen et al. ([1], Theorem 11). The proof of Bang-Jensen et al. can be easily modified to prove the following.

Theorem 6 *The recognition of perfect digraphs is co- \mathcal{NP} -complete.*

The preceding result can be obtained by a reduction of 3-SAT to recognition of non-perfect digraphs. This result is in contrast to the result of Chudnovsky et al. [2] which, together with the Strong Perfect Graph Theorem [3], states that the recognition of perfect graphs is in \mathcal{P} .

Note that the perfectness of digraphs does not behave as well as the perfectness of graphs in a second aspect: there is no analogon to Lovasz' Weak Perfect Graph Theorem [5]. A digraph may be perfect but its complement may be not perfect. An easy instance of this type is the directed 4-cycle \vec{C}_4 , which is not perfect, and its complement H , which is perfect.

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Hull number: $\{P_5, K_3\}$ -free graphs and reduction rules

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EXTENDED ABSTRACT

All graphs in this work are undirected, simple and loopless. Given a connected graph $G = (V, E)$, the closed interval $I[u, v]$ of any two vertices $u, v \in V$ is the set of vertices that belong to some u - v geodesic of G , i.e. some shortest (u, v) -path. For any $S \subseteq V$, let $I[S] = \bigcup_{u, v \in S} I[u, v]$. A subset $S \subseteq V$ is (*geodesically*) *convex* if $I[S] = S$. Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S . We say that a vertex v is *generated* by a set of vertices S if $v \in I_h[S]$. We say that S is a *hull set* of G if $I_h[S] = V$. The size of a minimum hull set of G is the *hull number* of G , denoted by $hn(G)$ [8].

It is known that computing $hn(G)$ is an NP-hard problem for bipartite graphs [2]. Several bounds on the hull number of triangle-free graphs are presented in [7]. In [6], the authors show, among other results, that the hull number of any P_4 -free graph, i.e. any graph without induced path with four vertices, can be computed in polynomial time.

It is well-known that:

Theorem 1 [3] *G is P_5 -free if, and only if, for every induced subgraph $H \subseteq G$ either H has a dominating clique or a dominating cycle on five vertices.*

We use this result to obtain the following one:

Theorem 2 *The hull number of a P_5 -free bipartite graph $G = (A \cup B, E)$ can be computed in linear time.*

More generally, we deduce that:

Theorem 3 *If G is a $\{P_5, K_3\}$ -free graph, then $hn(G)$ can be computed in polynomial time.*

In the second part of this work, we consider the version of the problem parameterized by the neighborhood diversity of the input graph. The *neighborhood diversity* of a graph G is the smallest k such that its vertex set can be partitioned into k sets S_1, \dots, S_k , such that any pair of vertices $u, v \in S_i$ satisfy $N(u) \setminus v = N(v) \setminus u$, for every $1 \leq i \leq k$. This parameter was proposed by Lampis [10] as a generalization of vertex cover and has been recently used to obtain FPT algorithms for several problems [9].

We make use of four reduction rules to obtain, from a graph G , a graph G^* having one vertex less and satisfying either $hn(G) = hn(G^*)$ or $hn(G) = hn(G^*) + 1$, according to the used rule. We then use these rules to obtain a fixed parameter tractable (FPT) algorithm, where the parameter is the neighborhood diversity of the input graph.

Given two graphs G and H , the *lexicographic product* $G \circ H$ is the graph whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and such that two vertices (g_1, h_1) and (g_2, h_2) are adjacent if, and only if, either $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$.

A characterization of the (geodesic) convex sets in the lexicographic product of two graphs is given in [1]. In [11] they study the pre-hull number for this product. There are also some results concerning the hull number of the cartesian and strong products of graphs [5, 4].

We use the reduction rules we present to characterize the hull number of the lexicographic product of any two graphs. We prove that:

Theorem 4 Let G be a connected graph and H be an arbitrary graph. Thus,

$$hn(G \circ H) = \begin{cases} 2, & \text{if } H \text{ is not complete;} \\ (|V(H)| - 1)|S(G)| + hn(G), & \text{if } H \text{ is complete.} \end{cases}$$

We finish this work by proposing two open questions:

Problem 5 For a fixed k , what is the computational complexity of determining $hn(G)$, for a P_k -free graph G ?

Problem 6 Is there any FPT algorithm to determine whether $hn(G) \leq k$, where k is the parameter?

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On the theta number of circular-cliques and some consequences

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EXTENDED ABSTRACT

In the seminal paper [4], Lovász introduced the so-called *theta number* $\vartheta(G)$ of a graph. On one hand, this number provides an approximation of $\omega(G)$ and of $\chi(G)$ since (this is the celebrated *Sandwich Theorem*)

$$\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G),$$

where \overline{G} stands for the complement of G . On the other hand, this number is the optimal value of a *semidefinite program* [5], and, as such, is computable in polynomial time with polynomial space encoding accuracy [5], [3]. In contrast, the computation of either of the clique number or the chromatic number is known to be NP-hard.

By definition, a graph G is *perfect*, if for every induced subgraph H , $\omega(H) = \chi(H)$ [2]. As $\vartheta(\overline{G}) = \chi(G)$ and $\chi(G)$ is an integer, we have

Theorem 1 (*Grötschel, Lovász and Schrijver*) [3] *For every perfect graph, the chromatic number is computable in polynomial time.*

The notions of clique and chromatic numbers and of perfect graphs have been refined using circular complete graphs. The *circular chromatic number* $\chi_c(G)$ of a graph G was first introduced by Vince in [6]. It is the minimum of the fractions k/d for which $G \rightarrow K_{k/d}$. Later, Zhu defined the *circular clique number* $\omega_c(G)$ of G to be the maximum of the k/d for which $K_{k/d} \rightarrow G$ and introduced the notion of a *circular perfect graph*, a graph with the property that every induced subgraph H satisfies $\omega_c(H) = \chi_c(H)$.

In this talk, we give the following closed formula for Lovász's theta number of the powers of cycle graphs C_k^{d-1} and of their complements, the circular complete graphs $K_{k/d}$:

Theorem 2 [1] *Let $d \geq 2$, $k \geq 2d$, with $\gcd(k, d) = 1$. Let, for $0 \leq n \leq d-1$,*

$$c_n := \cos\left(\frac{2n\pi}{d}\right), \quad a_n := \cos\left(\left\lfloor \frac{nk}{d} \right\rfloor \frac{2\pi}{k}\right).$$

Then

$$\vartheta(\overline{K_{k/d}}) = \frac{k}{d} \sum_{n=0}^{d-1} \prod_{s=1}^{d-1} \left(\frac{c_n - a_s}{1 - a_s} \right).$$

As a consequence, we establish that the circular-chromatic number of circular-perfect graphs is computable in polynomial time, which extends Theorem 1 from the chromatic number to the circular-chromatic number, and from perfect graphs to the superclass of circular-perfect graphs.

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Resolving sets for Johnson and Kneser graphs

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EXTENDED ABSTRACT

A vertex $x \in V(G)$ in a graph G is said to *resolve* a pair $u, v \in V(G)$ if $d_G(u, x) \neq d_G(v, x)$. A set $S \subseteq V(G)$ is *resolving* if any pair of vertices in the graph can be resolved by some vertex in S . If the set S is as small as possible, then it is called a *metric basis* and its cardinality $\beta(G)$ is the *metric dimension* of the graph G .

Metric bases and resolving sets were first introduced by Slater [8] and independently by Harary and Melter [4]. In his seminal paper, Slater mentioned as an application the location of a moving point in a graph by knowing the distances from the point to some sonar or Loran stations which have been conveniently stationed in the graph. Currently, it is possible to find many other applications in the literature. As instances, the study of resolvability in hypercubes is closely related with the coin weighting problem (see [7] as a survey), the strategies for the static Mastermind rely on resolving sets in Hamming graphs [2, 3], resolving sets in triangular, rectangular and hexagonal grids have been proposed to study digital images in [6], a method based on resolving sets for differentiating substances with the same chemical formulation is given in [1], etc.

Since the problem of computing the metric dimension of a graph is NP-complete (see [5]), many efforts have been focused on finding either exact values or tight bounds of the metric dimension for certain classes of graphs as wheels, trees, unicyclic graphs, cartesian products and Cayley digraphs among others. Here we are interested in Johnson and Kneser graphs.

A *Kneser graph* $K(n, k)$ (where $n > k$) has the k -subsets of the n -set $[n] = \{1, \dots, n\}$ as vertices, and edges connecting disjoint sets. As an example, the graph $K(5, 2)$ is the Petersen graph shown in Figure 4(a). Like Kneser graphs, the vertices of a *Johnson graph* $J(n, k)$, with $n > k$, are the k -subsets of $[n]$ but its edges are determined by intersections of size $k - 1$ (see Figure 4(b)).

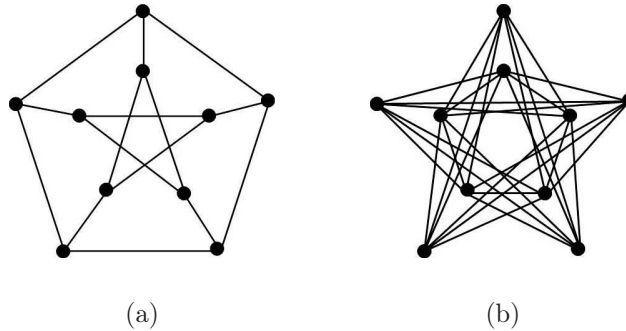


Figure 4: (a) The Kneser graph $K(5, 2)$, (b) The Johnson graph $J(5, 2)$

Our goal is to explore the relationship between several combinatorial and geometric structures with the resolvability in Kneser and Johnson graphs. Particularly, we will use partitions, incidence matrices, symmetric designs, Steiner systems, partial geometries and

toroidal grids. Although we will like to emphasize those relationships, in particular some of those constructions provide tight lower bounds on the metric dimension of $J(n, k)$ and/or $K(n, k)$.

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On some facility location problem

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EXTENDED ABSTRACT

Introducion. *Facility location problems* (FLPs) focus on finding appropriate locations for a company or a government to serve its clients. It can either be a minimization problem (when considering costs) or a maximization problem (when considering service). These problems have gathered many attention during the second half of the twentieth century (see the book of Daskin [2] for historical review and the article of Klose and Drexl [3] for extensive bibliography).

We focus on a specific FLP. Given a directed graph $D = (V, A)$ with weights on arcs and vertices, you must first select a set of nodes (the facility locations) called *centers* and then affect each remaining vertex to exactly one center (*this* client is served by *that* center) such that there is an arc going from the affected vertex to the chosen center. The weight of each vertex corresponds to the expected revenue minus the cost of building a center on that location. The weight of an arc consists of the expected revenue of this affectation minus any generated cost. The aim is to maximize the revenue.

This problem can easily be described as 0-1 integer linear program and has attracted focus from polytope's scientists as Baïou and Barahona [1].

From digraphs to graphs. From the input digraph $D = (V, A)$, we seek to find the affectation arcs. This set of arcs must verify:

- (i) No two arcs have same vertex as tail (a client would be served by two facilities).
- (ii) No vertex can be the head of an arc and the tail of another (it is either a client or a center).

Now, call *independent* two arcs from D that have no common extremity or have same heads. And define its so-called *line graph* as $G = (A, E)$ having an vertex for each arc in D and edges between two arcs if they are not independent. We call such a graph a *facility-location graph*. The original problem on D is then closely related to the problem of finding a stable set of maximum weight in G (with an appropriate choice of weights on G 's vertices).

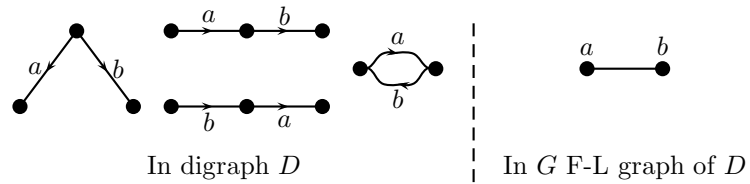


Figure 5: Rules for building facility-location graphs

Figure 6 shows an exemple for a given digraph.

However, not every undirected graph is a facility-location graph. For exemple if you add an edge between vertices b and e in the undirected graph of Figure 6, the obtained graph has no antecedent through our transformation.

Recognizing facility-location graphs. We have been investigating this question. We obtained several results.

Proposition 1 *Let G be an undirected graph and e an edge connecting two vertices of degree exactly 2. Then G is a facility location graph if and only if $G \setminus e$ is a facility location graph.*

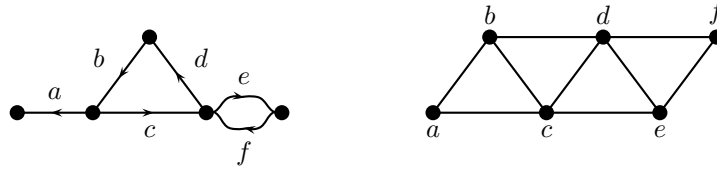


Figure 6: A digraph and its facility-location graph

With this last proposition, we may peel off every ear of G , and get the following result.

Proposition 2 *Given G an triangle-free graph, it is a facility location graph if and only if, after having peeled the ears off, each of the remaining connected components has at most one cycle.*

The main question behind those lines is the next one.

Question 3 *Is it easy to recognize facility location graphs ?*

In this talk, we will elaborate about this last question.

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Arc-disjoint paths in decomposable digraphs

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EXTENDED ABSTRACT

Notation follows [2]. A digraph is called **semicomplete** when there is an arc between every two distinct vertices. A semicomplete digraph without 2-cycles is called a **tournament**. A digraph $D = (V, A)$ is **locally semicomplete** if the set of out-neighbours and the set of in-neighbours of v both induce semicomplete digraphs for every $v \in V$. A digraph is **quasi-transitive** if the presence of the arcs xy, yz implies that there is an arc between x and z (which may be xz or zx or both).

Let $D = (V, A)$ be a digraph and let $s_1, \dots, s_k, t_1, \dots, t_k$ be a collection of (not necessarily distinct) vertices of D . A **weak k -linkage** from (s_1, \dots, s_k) to (t_1, \dots, t_k) is a collection of k arc-disjoint paths P_1, \dots, P_k such that P_i is an (s_i, t_i) -path or a proper cycle containing $s_i = t_i$ for each $i \in [k]$. The **WEAK k -LINKAGE PROBLEM** is the following. Given a digraph $D = (V, A)$ and not necessarily distinct vertices $s_1, \dots, s_k, t_1, \dots, t_k$; decide whether D contains a weak k -linkage from (s_1, \dots, s_k) to (t_1, \dots, t_k) . It is well-known that the weak k -linkage problem is NP-complete already when $k = 2$ [3].

Until very recently, the only non-trivial class of digraphs for which the weak k -linkage problem was known to be polynomial for every fixed k was the class of acyclic digraphs, for which there was an algorithm [3] due to Fortune, Hopcroft and Wyllie.

Theorem 1 [3] *The weak k -linkage problem is polynomially solvable for every fixed k when the input is an acyclic digraph.*

Even for the class of tournaments a polynomial algorithm was only known for the case of $k = 2$ [1]. The following results due to Fradkin and Seymour change that drastically.

Theorem 2 (Fradkin-Seymour) [4] *The weak k -linkage problem is polynomial for every fixed k , when we consider digraphs that are obtained from a semicomplete digraph by replacing some arcs with multiple copies of those arcs and adding any number of loops.*

Theorem 3 (Fradkin-Seymour) [4] *For every natural number α the weak k -linkage problem is polynomial for every fixed k , when we consider digraphs with independence number at most α .¹*

Let $D = (V, A)$ be a digraph and $O = v_1, \dots, v_n$ an ordering of its vertices. We say that O has **cutwidth** at most θ if for all $j \in \{2, 3, \dots, n\}$ there are at most θ arcs uv with $u \in \{v_1, \dots, v_{j-1}\}$ and $v \in \{v_j, \dots, v_n\}$ and we say that D has cutwidth at most θ if there exists an ordering O of $V(D)$ which has cutwidth at most θ . The minimum θ such that D has cutwidth at most θ is called the **cutwidth** of D and is denoted by $cw(D)$. The following result is implicitly stated in [4].

Theorem 4 (Fradkin-Seymour) [4] *For every natural number θ the weak k -linkage problem is polynomial for every fixed k , when we consider digraphs with cutwidth at most θ .*

Definition 5 *If S is a digraph on s vertices v_1, v_2, \dots, v_s and H_1, \dots, H_s are disjoint digraphs, we denote by $D = S[H_1, \dots, H_s]$ the digraph which we obtain by replacing the i th vertex v_i of S by H_i (we say that we **blow up** v_i to H_i) and adding all possible arcs from $V(H_i)$ to $V(H_j)$ whenever ij is an arc of S . A digraph D is **decomposable** if $D = S[H_1, \dots, H_s]$,*

¹The independence number of a digraph D is the cardinality of the largest set $I \subseteq V(D)$ such that $A(D(I)) = \emptyset$.

for some digraph S , with $s = |V(S)| \geq 2$ and some choice of disjoint digraphs H_1, \dots, H_s . Let Φ be a class of digraphs. We say that D is **totally Φ -decomposable** if either $D \in \Phi$ or $D = S[H_1, \dots, H_s]$ is decomposable with $S \in \Phi$ and H_i totally Φ -decomposable, for $i = 1, \dots, s$. We call this decomposition along with total Φ -decompositions of each H_i a **total Φ -decomposition** of D .

Given a digraph D and a non negative integer c , let $D(c)$ denote the set of digraphs that can be obtained from D by first adding any number of arcs parallel to the already existing ones and then blowing up b vertices, with $0 \leq b \leq c$, to digraphs of size less than or equal to c each.

Definition 6 We say that a class of digraphs Φ is **bombproof** if there exists a polynomial algorithm \mathcal{A}_Φ to find a total Φ -decomposition of every totally Φ -decomposable digraph and, for every integer c , there exists a polynomial algorithm² \mathcal{B}_Φ to decide the weak k -linkage problem for the class

$$\Phi(c) := \bigcup_{D \in \Phi} D(c).$$

Our main result, whose proof uses Theorems 3 and 4 is the following.

Theorem 7 For every fixed k there exists a polynomial algorithm for the weak k -linkage problem for the totally Φ -decomposable digraphs.

In order to prove this theorem we needed the following stronger version dealing with digraphs that can be obtained from totally Φ -decomposable digraphs by deleting a bounded set of arcs in a controlled way.

Theorem 8 There is a polynomial algorithm \mathcal{M} which takes as input a 5 tuple $[D, k, k', \Pi, F]$, where Φ is bombproof, D is a totally Φ -decomposable digraph, k, k' are natural numbers with $k' \leq k$, Π is a list of k' terminal pairs and $F \subseteq A(D)$ is a set of arcs satisfying

$$\begin{aligned} d_F^-(v), d_F^+(v) &\leq k - k' \quad \text{for all } v \in V(D). \\ |F| &\leq (k - k')2k \end{aligned} \tag{1}$$

and decides whether $D \setminus F$ contains a weak Π -linkage.

Using Theorem 7 and a number of structural results on these classes, we prove the following.

Theorem 9 The weak k -linking problem is polynomially solvable for every fixed k in digraphs that are either quasi-transitive, locally semicomplete or extended semicomplete.

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²Note that the running time of \mathcal{B}_Φ may depend heavily on c .

On Graphs with Maximal Independent Sets of Few Sizes, Minimum Degree at least 2, and Girth at least 7

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EXTENDED ABSTRACT

We consider finite, simple, and undirected graphs. For a graph G , the vertex set and the edge set are denoted $V(G)$ and $E(G)$, respectively. For a vertex u of G , the neighbourhood, the closed neighbourhood, and the degree are denoted $N_G(u)$, $N_G[u]$, and $d_G(u)$, respectively. For some set U of vertices of G , let $N_G[U] = \bigcup_{u \in U} N_G[u]$.

A set I of vertices of a graph G is independent if no two vertices in I are adjacent. An independent set I of G is maximal if every vertex u in $V(G) \setminus I$ has a neighbour in I . An independent set I of G is maximum if G has no independent set J with $|J| > |I|$.

The minimum length of a cycle in a graph G is the girth of G .

In [7] Plummer defines a graph to be *well-covered* if all its maximal independent sets have the same size. Generalizing this concept, Finbow, Hartnell, and Nowakowski [4] define, for every $r \in \mathbb{N}$, the set \mathcal{M}_r as the set of graphs that have maximal independent sets of exactly r different sizes. With this notation, \mathcal{M}_1 contains exactly all well-covered graphs.

The structure of well-covered graphs or more generally of the graphs in \mathcal{M}_r is far from completely understood [5, 8]. Many results on \mathcal{M}_r concern graphs without small cycles. In [3] Finbow, Hartnell, and Nowakowski prove that C_7 is the only well-covered graph of minimum degree at least 2 and girth at least 6. Similarly, Finbow, Hartnell, and Whitehead [4] prove that C_8 , C_9 , C_{10} , C_{11} , and C_{13} are the only graphs in \mathcal{M}_2 of minimum degree at least 2 and girth at least 8 and, for $r \geq 4$, Hartnell and Rall [6] prove that every graph in \mathcal{M}_r of minimum degree at least 2 and girth at least $6r - 6$ is a cycle. For $r \in \{2, 3\}$, the set \mathcal{M}_r contains graphs of minimum degree at least 2 and girth $6r - 5$ that are not cycles [6].

In the present paper we study connected graphs in \mathcal{M}_r that are of minimum degree at least 2 and girth at least 7. Our motivation was the conjecture that, for $r = 2$, there are only finitely many such graphs all of which are of maximum degree at most 4. As our main result, we prove, for every r , that there are only finitely many such graphs of bounded maximum degree. Furthermore, we prove several results restricting the degrees of such graphs.

For further related results concerning \mathcal{M}_r and (odd) girth conditions, refer to [1, 2, 9].

For $r \geq 1$, let $\mathcal{M}_{\leq r} = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_r$, that is, $\mathcal{M}_{\leq r}$ is the set of graphs that have independent sets of at most r different sizes. We begin with our main result.

Theorem 1 *For $r \geq 1$ and $\Delta \geq 3$, every graph in $\mathcal{M}_{\leq r}$ that is of minimum degree at least 2, maximum degree at most Δ , and girth at least 7 has order at most*

$$\frac{\Delta}{\Delta - 2} \left((\Delta - 1)^{36(r+1)^2 + 42(r+1) + 5} - 1 \right).$$

In view of Theorem 1 conditions that imply a bounded maximum degree of the considered graphs are of interest. Our remaining results illustrate different conditions that allow to restrict the vertex degrees.

Theorem 2 *For $r \geq 0$ and $s \geq 0$, every graph in $\mathcal{M}_{\leq r}$ that is of minimum degree at least 2 and girth at least 7, and in which for every vertex u , there are at most s cycles of length 7 that intersect in u , has maximum degree at most $\binom{r}{2} + 2s + 1$.*

Theorem 3 implies that for every $r \geq 1$, the graphs in $\mathcal{M}_{\leq r}$ of minimum degree at least 2 and girth at least 8, have bounded maximum degree. Hence, by Theorem 1, there are only finitely many such graphs. For $r \in \{1, 2\}$, this was known [3, 4].

Our next result limits the number of neighbours of minimum degree for every vertex.

Proposition 3 *If $r \geq 0$ and G is a graph in $\mathcal{M}_{\leq r}$ of minimum degree at least 2 and girth at least 7, then every vertex in G has at most $\left(\binom{r}{2} + 1\right)^2$ neighbours of degree 2.*

Our next result is a variation of Proposition 3. It shows that vertices of large degree cannot have too many vertices of considerably smaller degree within distance 3.

Proposition 4 *Let $r \geq 0$ and let G be a graph in $\mathcal{M}_{\leq r}$ of minimum degree at least 2 and girth at least 7.*

If u is a vertex of G and N is a set of d_1 neighbours of u such that every vertex in N has degree at most d_2 and for every vertex v in N and every vertex w in $N_G(v) \setminus \{u\}$, there is a vertex z in $N_G(w) \setminus \{v\}$ of degree at most d_3 , then

$$\frac{d_1}{(d_2 - 1)(d_3 - 1) + 1} - 1 \leq \binom{r}{2}.$$

In our next result we consider graphs of girth at least 6 that do not contain cycles of length 7. While we cannot bound the maximum degree in this case, we can at least bound the number of different vertex degrees.

Proposition 5 *For $r \geq 0$, every graph G in $\mathcal{M}_{\leq r}$ that is of minimum degree at least 2, girth at least 6, and does not contain a cycle of length 7, has at most $\binom{r}{2}$ different vertex degrees.*

We close by repeating our conjecture:

Conjecture 6 *There are only finitely many graphs in \mathcal{M}_2 that are of minimum degree at least 2 and girth at least 7. Furthermore, all these graphs are of maximum degree at most 4.*

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Generalizing $[2, b]$ -Factors

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EXTENDED ABSTRACT

A factor of a graph G is a spanning subgraph of G . For two integers a and b such that $0 \leq a \leq b$, an $[a, b]$ -factor is a factor in which the degree of each vertex is between a and b . It can also be defined in terms of graph partition by vertex-disjoint components whose vertices are of degree between a and b . Factors, particularly $[a, b]$ -factors, have been studied in lots of papers and several sufficient conditions have been established for the existence of factors having some specific properties, most often on degrees.

In [2], Kouider and Lonc showed that if a and b are two integers such that $b \geq a + 1$ and if G is a graph with minimum degree δ and independence number α verifying:

$$\alpha \leq \begin{cases} \frac{4b(\delta-a+1)}{(a+1)^2} & \text{if } a \text{ is odd} \\ \frac{4b(\delta-a+1)}{a(a+2)} & \text{if } a \text{ is even,} \end{cases} \quad \text{then } G \text{ has an } [a, b]\text{-factor.}$$

In particular, for $a = 2$, we obtain that if $\alpha \leq \frac{b(\delta-1)}{2}$ then G has a $[2, b]$ -factor. But what happens if $\alpha > \frac{b(\delta-1)}{2}$? Let us consider a partition of G by vertex-disjoint components so that each component is a vertex, an edge or with its vertices of degree between 2 and b . Clearly, such a partition with neither edges nor vertices is nothing but a $[2, b]$ -factor. We investigate the relationship between the independence number, the minimum degree and the number of components that are edges or vertices in such a partition in order to obtain an upper bound for the number of these components. The work has already been done for $b = 2$ in [1]. Here we propose to extend it to larger b and obtain the following result:

Let b be an integer such that $b \geq 4$ and G a graph of minimum degree δ , independence number α such that $\alpha > \frac{b(\delta-1)}{2}$ and without isolated vertices. Then G possesses a partition with vertex-disjoint components such that each component C is a vertex, an edge or verifies $2 \leq d_C(x) \leq b$ for all x in C and such that this partition contains at most $\alpha - \lfloor \frac{b(\delta-1)}{2} \rfloor$ components that are edges or vertices.

Furthermore, the bound given above is best possible.

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A Unified FPT Algorithm for Width of Partition Functions

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EXTENDED ABSTRACT

Background

The notion of *treewidth* is central in the theory of the Graph Minors developed by Robertson and Seymour [1]. Roughly, the treewidth of a graph measures how close a graph is to a tree.

The notion of treewidth plays an important role in the domain of algorithmic computational complexity. Indeed, many graph theoretical problems that are NP-complete in general are tractable when input graphs have bounded treewidth. In this context, many *Fixed Parameter Tractable* (FPT) algorithms have been designed to solve problems like Hamiltonian Circuit, Independent Set, Graph Coloring, etc. More generally, the celebrated theorem of Courcelle states that any monadic second-order graph properties can be decided in linear time in the class of graphs of bounded treewidth [2]. Typically, these algorithms are based on dynamic programming on a given tree-decomposition of a graph.

Thus, an important challenge consists in computing tree-decompositions of graphs with small width. This problem is NP-complete [3] and special interest has been directed toward special graph classes [4, 5].

An immediate consequence from the seminal work on Graph Minors [1] of Robertson and Seymour is the existence of a polynomial-time algorithm to decide whether a graph has treewidth at most k , where k is a fixed parameter. Hence, in [6], Bodlaender and Kloks design a linear time algorithm such that for k and k' fixed, and given a n -node graph G , a tree-decomposition of width at most k' of G , the algorithm decides if $tw(G) \leq k$. Unfortunately, with a constant more than exponential in k and k' . In the last decades, analogous algorithms have been designed for several other width parameters [6, 7, 8].

Both pathwidth and treewidth have also a nice theoretical-game interpretation [9]. In [10], Fomin *et al.* introduce a variant of these games, called *non-deterministic graph searching*, establishing a link between pathwidth and treewidth. The q -branched tree-decomposition which is a rooted tree-decomposition where $\infty \geq q \geq 0$ is the maximum number of vertices in any path from the root with at least two children.

Fomin *et al.* prove that calculating this width parameter is NP-hard for any $q \geq 0$ and n -node graph G [10]. Prior to this work, no explicit FPT algorithm for this problem was known.

Lastly, for the special tree decomposition [11], used in the recognition some properties of graphs that can be translated into second order monadic order, there is also no explicit known algorithm prior to this work.

Our results

Our work aims to unify and generalize the FPT algorithms for computing various decompositions of graphs. As a particular application, our algorithm decides in linear time if the q -limited width parameter of a graph G is at most k , $q \geq 0$ and $k \geq 1$ fixed.

In order to generalize the algorithm of [6], we use the notions of partition function and partitioning tree defined in [12]. Given a finite set A , a partition function Φ for A is a function from the set of partitions of A into the integers. A partitioning-tree of A is a tree T together with a one-to-one mapping between A and the leaves of T . The Φ -width of T is the maximum $\Phi(\mathcal{P})$ of A defined by the internal vertices of T . Partition functions are a

unified view for a large class of width parameters like treewidth, pathwidth, branchwidth, etc.

In this work, we extend the definition of Φ -width to the one of q -branched Φ -width of a set A . Then, we use the framework of [6] applied to the notions of partition functions and partitioning-tree in order to design a unified linear-time algorithm that decides if a finite set has q -branched Φ -width at most k . Again, $q \geq 0$ and $k \geq 1$ are fixed parameters.

We propose an algorithm such that, for any k and q fixed parameters, and any partition function Φ satisfying some properties, our algorithm decides in time $O(|A|)$ if a finite set A has q -branched Φ -width at most k . Since the mentioned width parameters can be defined in terms of Φ -width, our algorithm unifies the works in [6, 8, 7]. Moreover, our algorithm generalizes the previous algorithms since it is not restricted to width-parameters of graphs. Finally, it provides the first explicit linear-time algorithm that decides if a graph G has a q -branched tree-decomposition or a special tree-decomposition of width k , for any $k \geq 1$, $q \geq 0$ fixed.

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Adjacent vertex-distinguishing edge coloring of graphs with bounded maximum degree

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EXTENDED ABSTRACT

In the following we only consider simple graphs with no isolated edges. A proper edge coloring of a graph $G = (V, E)$ is an assignment of colors to the edges of the graph such that two adjacent edges do not use the same color. An *adjacent vertex-distinguishing k -edge coloring*, or *k -avd-coloring* for short, of a graph G is a proper edge coloring of G using at most k colors such that, for every pair of adjacent vertices u, v , the set of colors of the edges incident to u differs from the set of colors of the edges incident to v . Adjacent vertex-distinguishing colorings are also known as *adjacent strong edge coloring* and *1-strong edge coloring*. We denote by $\chi'_{avd}(G)$ the *avd-chromatic number* of G , which is the smallest integer k such that G can be k -avd-colored.

Zhang *et al.* completely determined the avd-chromatic number for paths, cycles, trees, complete graphs, and complete bipartite graphs [5]. Moreover, they proposed the following conjecture [5], where $\Delta(G)$ denotes the maximum degree of G :

Conjecture 1 *If G is a connected graph with at least 6 vertices, then $\chi'_{avd}(G) \leq \Delta(G) + 2$.*

Balister *et al.* [1] proved Conjecture 1 for graphs with maximum degree three and for bipartite graphs.

Let $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$ be the maximum average degree of the graph G , where $V(H)$ and $E(H)$ are the sets of vertices and edges of H , respectively.

Wang and Wang [3] linked maximum average degree and avd-colorings, and their results were further improved by Hocquard and Montassier [2]:

Theorem 2 [3, 2] *Let G be a graph with maximum degree $\Delta(G)$.*

1. *If $\Delta(G) \geq 3$ and $\text{mad}(G) < 3$, then $\chi'_{avd}(G) \leq \Delta(G) + 2$.*
2. *If $\Delta(G) \geq 3$ and $\text{mad}(G) < 3 - \frac{2}{\Delta(G)}$, then $\chi'_{avd}(G) \leq \Delta(G) + 1$.*

Two main questions arise from these partial results: can this threshold of 3 as an upper-bound on $\text{mad}(G)$ be reached with a sufficiently large lower-bound on $\Delta(G)$ in the case of Theorem 2.2, and broken in the case of Theorem 2.1?

We answer positively to these questions with Theorem 3.1, and prove more precisely that there is no threshold in the case of Theorem 2.2 (and thus in the case of Theorem 2.1). Theorem 3.2-3 are refinements of it for small upper-bounds on $\text{mad}(G)$.

Theorem 3 *Let G be a graph with maximum degree $\Delta(G)$ and maximum average degree $\text{mad}(G)$.*

1. *For all $m \geq \frac{5}{2}$, if $\Delta(G) \geq 16m^2$ and $\text{mad}(G) < m$, then $\chi'_{avd}(G) \leq \Delta(G) + 1$.*
2. *If $\Delta(G) \geq 7$ and $\text{mad}(G) < 3$, then $\chi'_{avd}(G) \leq \Delta(G) + 1$.*
3. *If $\Delta(G) \geq 3$ and $\text{mad}(G) < 3 - \frac{1}{\Delta(G)-1}$, then $\chi'_{avd}(G) \leq \Delta(G) + 1$.*

Theorem 3 is proved using discharging methods. The proof of Theorem 3.1 is heavily inspired from a beautiful proof by Woodall [4] for a similar statement in the case of list edge coloring with Δ colors. Note that list edge coloring is similarly conjectured to be always possible with $\Delta + 1$ colors. The main idea in the proof of Theorem 3.2 is that a minimal (minimal in the lexico-graphic order) counter-example contains no cycle in which every edge is incident to at least one vertex of degree 2. In the proof of Theorem 3.3, we consider a tree-like configuration of unbounded size that contains all the locally problematic cases, and we prove that it cannot be problematic on the whole graph if the upper-bound on the maximum average degree is respected.

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Graphs with $\text{mad} < 3$ and $\Delta \geq 17$ are list 2-distance $(\Delta + 2)$ -colorable

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EXTENDED ABSTRACT

All the graphs considered here are simple and finite. A *2-distance k -coloring* of a graph G is a coloring of the vertices of G with k colors such that two vertices that are adjacent or have a common neighbor receive distinct colors. We define $\chi^2(G)$ as the smallest k such that G admits a 2-distance k -coloring. A generalization of the 2-distance k -coloring is the *list 2-distance k -coloring*, where instead of having the same list of k colors for the whole graph, every vertex is assigned some set of k colors and has to be colored from it. We define $\chi_\ell^2(G)$ as the smallest k such that G admits a list 2-distance k -coloring of G for any list assignment. Obviously, 2-distance coloring is a sub-case of list 2-distance coloring (where the same color list is assigned to every vertex), so for any graph G , $\chi_\ell^2(G) \geq \chi^2(G)$.

The study of $\chi^2(G)$ on planar graphs was initiated by Wegner in 1977 [7], and has been actively studied because of his conjecture, stated below. The *maximum degree* of a graph G is denoted $\Delta(G)$.

Conjecture 1 (Wegner [7]) *If G is a planar graph, then:*

- $\chi^2(G) \leq 7$ if $\Delta(G) = 3$
- $\chi^2(G) \leq \Delta(G) + 5$ if $4 \leq \Delta(G) \leq 7$
- $\chi^2(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ if $\Delta(G) \geq 8$

Note that any graph G satisfies $\chi^2(G) \geq \Delta(G) + 1$. Indeed, if we consider a vertex of maximal degree and its neighbors, they form a set of $\Delta(G) + 1$ vertices, any two of which are adjacent or have a common neighbor. Hence at least $\Delta(G) + 1$ colors are needed for a 2-distance coloring of G . It is therefore natural to ask when this lower bound is reached. For that purpose, we can study, as suggested by Wang and Lih [6], what conditions on the sparseness of the graph can be sufficient to ensure the equality holds. A first measure of the sparseness of a planar graph is its girth. The *girth* of a graph G , denoted $g(G)$, is the length of a shortest cycle.

Conjecture 2 (Wang and Lih [6]) *For any integer $k \geq 5$, there exists an integer $D(k)$ such that for every planar graph G verifying $g(G) \geq k$ and $\Delta(G) \geq D(k)$, $\chi^2(G) = \Delta(G) + 1$.*

Conjecture 2 was proved by Borodin, Ivanova and Noestroeva [3, 4] to be true for $k \geq 7$, even in the case of list-coloring, and false for $k \in \{5, 6\}$.

Dvořák, Král, Nejedlý and Škrekovski [5] proved that it is off by just one for $k = 6$, i.e. for a planar graph G with girth 6 and sufficiently large $\Delta(G)$, $\chi^2(G) \leq \Delta(G) + 2$. They also conjectured that the same holds for planar graphs with girth 5, but this remains open. Borodin and Ivanova [1, 2] improved the corresponding bound for graphs of girth 6, and extended it to list-coloring.

Theorem 3 (Borodin and Ivanova [1]) *Every planar graph G with $\Delta(G) \geq 18$ and $g(G) \geq 6$ admits a 2-distance $(\Delta(G) + 2)$ -coloring.*

Theorem 4 (Borodin and Ivanova [2]) *Every planar graph G with $\Delta(G) \geq 24$ and $g(G) \geq 6$ admits a list 2-distance $(\Delta(G) + 2)$ -coloring.*

We improve the previous two theorems as follows.

Theorem 5 *Every planar graph G with $\Delta(G) \geq 17$ and $g(G) \geq 6$ admits a list 2-distance $(\Delta(G) + 2)$ -coloring.*

Another way to measure the sparseness of a graph is through its maximum average degree as defined below. The *average degree* of a graph G , denoted $\text{ad}(G)$, is $\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|}$. The *maximum average degree* of a graph G , denoted $\text{mad}(G)$, is the maximum of $\text{ad}(H)$ over all subgraph H of G . Using this measure, we, in fact, prove a more general theorem than Theorem 5.

Theorem 6 *Every graph G with $\Delta(G) \geq 17$ and $\text{mad}(G) < 3$ admits a list 2-distance $(\Delta(G) + 2)$ -coloring.*

Euler's formula links girth and maximum average degree in the case of planar graphs, as it is easy to check that for any planar graph G , $(\text{mad}(G) - 2)(g(G) - 2) < 4$. Thus, planar graphs of girth at least 6 have a maximum average degree smaller than 3, and Theorem 5 is a corollary of Theorem 6.

To prove Theorem 6, we use a global discharging method, that is, a discharging method where some forbidden configurations have unbounded size and where the weight can travel arbitrarily far.

An *injective k -coloring* of G is a (not necessarily proper) coloring of the vertices of G with k colors such that no vertex has two neighbors with the same color, or, in other words, such that two vertices that have a common neighbor receive distinct colors. A 2-distance k -coloring is also an injective coloring, but the reverse is not true. The list version of this coloring is a *list injective k -coloring* of G .

Some results on 2-distance coloring have their counterpart on injective coloring with one less color, and it is the case of Theorems 3 and 4. It happens that the proof of Theorem 6 also works with close to no alteration for list injective coloring, thus yielding a proof that every graph G with $\Delta(G) \geq 17$ and $\text{mad}(G) < 3$ admits a list injective $(\Delta(G) + 1)$ -coloring.

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Dominator chromatic number of a tree

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EXTENDED ABSTRACT

Let $G = (V, E)$ be a simple graph. A vertex in a graph G is said to *dominate* itself and every vertex adjacent to it. A subset D of V is a *dominating set* of G , if every vertex not in D is adjacent to at least one vertex in D . The *domination number*, $\gamma(G)$, is the minimum cardinality among all the dominating sets of G . The *dominator coloring* of a graph G is an assignment of colors to the vertices of G such that it is a proper coloring and every vertex dominates all vertices of at least one color class (possibly its own class if it's alone sharing the color). The minimum number of colors required for a dominator coloring of G is called the *dominator chromatic number* of G and is denoted by $\chi_d(G)$. Recently, Arumugam & al. [1] showed that, unless $P = NP$, χ_d cannot be computed in polynomial time on bipartite, planar or split graphs. A narrow bound for nontrivial trees T given by Gera [3, 4] shows that $\chi_d(T)$ is bounded below by $\gamma(T) + 1$ and above by $\gamma(T) + 2$. Moreover, a characterization of trees T with $\chi_d(T) = \gamma(T) + 1$ was given in [2]. However this result, as it is, is not easy to check for any tree, and therefore the problem of computing the exact value of the dominator coloring number of a tree remains open.

We provide a polynomial time algorithm computing χ_d of nontrivial trees. Before presenting our results, we need to introduce some notations and definitions. The *private neighborhood* of a vertex v of X with respect to X is $Pn(v, X) = N[v] \setminus N[X \setminus \{v\}]$. The set of stems of G is denoted by $S(G) = S$. A *star* R_k is a nontrivial tree of order k with at least $k - 1$ leaves. Let $V_1, V_2, \dots, V_{\chi_d}$ be the color classes of a minimum dominator coloring of G . A vertex $v \in V_i$ is called *solitary* if $|V_i| = 1$. We denote by C_P the set of color classes containing solitary vertices, by C_S the set of color classes, where each of them contains at least two vertices and is dominated by at least one vertex, and by C_G the set of color classes, where each of them contains at least two vertices and is not dominated by any vertex. Clearly, C_P , C_S and C_G form a partition of $\{V_1, V_2, \dots, V_{\chi_d}\}$. Also, let A be the set of all solitary vertices and B the set of all vertices belonging to color classes in C_G . Clearly, $|C_P| = |A|$. We denote by x_R , a vertex dominating the color class $R \in C_S$ and let $DS = \{x_R \in V : R \in C_S\}$. A *vertex cover* in a graph G is a set of vertices that covers all edges of G . Thus D is a vertex cover in G if and only if $V - D$ is an independent set. The *vertex cover number*, $\alpha_0(G)$, is the minimum cardinality of a vertex cover in G . A set $X \subseteq V(G)$ is a *packing set* of G if $N[x] \cap N[y] = \emptyset$ holds for any two distinct vertices $x, y \in X$.

Volkman gives in [5], a polynomial time characterization of trees with equal domination and vertex cover numbers. The following result shows that trees T with equal domination and vertex cover numbers satisfy $\chi_d(T) = \gamma(T) + 1$.

Proposition 1 *Let T be a nontrivial tree with $\gamma(T) = \alpha_0(T)$. Then $\chi_d(T) = \gamma(T) + 1$.*

Due to space constraints, proofs have been deferred to a longer version of the paper.

Note that the converse of Proposition 1 is not true in general. To see consider the tree T in Figure 7. Then T has $\chi_d(T) = 5$ but $\gamma(T) = 4 < \alpha_0(T) = 5$.

Let $Q = V \setminus N[S]$ and $Q^* = \{x \in V \setminus S : x \in N(S) \setminus N(Q)\}$. Clearly $L \subset Q^*$. We shall prove:

Theorem 2 *Let T be a nontrivial tree. Then, $\chi_d(T) = \gamma(T) + 1$ if and only if either T is a star or T is a tree such that Q is a packing set and Q^* is independent.*

To prove Theorem 2, next results are necessary. The first lemma establishes that without loss of generality, we may consider that leaves share the same color.

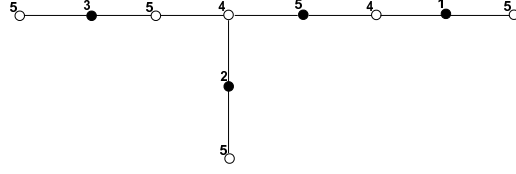


Figure 7: A tree T with $\chi_d(T) = \gamma(T) + 1$ and $\gamma(T) \neq \alpha_0(T)$.

Lemma 3 ([2]) *Every tree T of order at least three admits a minimum dominator coloring such that all leaves have the same color.*

The next two lemmas give some properties when minimum dominator coloring reaches the lower bound.

Lemma 4 ([2]) *Let T be a nontrivial tree different from a star. If $\chi_d(T) = \gamma(T) + 1$, then for every minimum dominator coloring of T such that all leaves have the same color, we have:*

- a) $|C_G| = 1$.
- b) $A \cup DS$ is a minimum dominating set of T . Furthermore, $A \cap DS = \emptyset$.
- c) Every color class $R \in C_S$ is dominated by a vertex $x_R \in B$, ie. $DS \subseteq B$.

Lemma 5 *Let T be a nontrivial tree different from a star. If $\chi_d(T) = \gamma(T) + 1$, then for every minimum dominator coloring of T such that all leaves have the same color, every vertex of $V \setminus (A \cup B)$ is adjacent to at least one stem.*

We show that in a dominator coloring with $\gamma(T) + 1$ colors of a tree T of order at least three such that all leaves of T have the same color, the vertices of DS are isolated in the subgraph induced by $A \cup DS$.

Lemma 6 *Let T be a nontrivial tree different from a star. If $\chi_d(T) = \gamma(T) + 1$, then for every dominator coloring with $\chi_d(T)$ colors such that all leaves have the same color, we have $N(DS) \subseteq V \setminus (A \cup B)$.*

Moreover, in such coloring, we may consider that each solitary vertex is a stem.

Lemma 7 *Let T be a nontrivial tree with $\chi_d(T) = \gamma(T) + 1$. Then there exists a minimum dominator coloring such that all leaves have the same color and every solitary vertex is a stem.*

Based on Theorem 2, we give a polynomial time algorithm computing the dominator chromatic number for nontrivial trees.

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Supergeodetic sets in graphs

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EXTENDED ABSTRACT

Although convexity notions in graph theory have been traditionally studied from an abstract point of view, very recently one can find in the literature a growing interest in computational aspects of the field. Particularly, the geodetic closure is analysed in [1, 2] as a tool for reconstructing the entire graph or a certain vertex subset from only a few points in it, similarly to the Euclidean case. Recall that, given a graph $G = (V, E)$ and a subset of vertices $S \subseteq V$, the geodetic closure $I(S)$ of S in G is defined to be the smallest set containing all the internal vertices from all the shortest paths among the vertices in S . A subset $S \subset V$ is called geodetic if $V(G) = I(S)$ and the cardinality of a minimum geodetic set is called the geodetic number $g(G)$ of G .

In general, to compute $I(S)$, you need to find all shortest paths between all vertices in S . However, it is well-known that for some cases (trees or complete graphs for instance) it is not necessary to explore all the pairs of vertices. Following this idea, the aim of this work is to determine those families of graph for which the computation of the geodetic closure is easier than the general case, in other words those graphs which are supergeodetic in a sense.

Given a graph G , a subset $S \subset V$ is called *supergeodetic* if there exists a distinguished vertex $u \in S$ such that $V = \bigcup_{v \in S} I(u, v)$. In other words, any vertex of G lie in some geodesic between u and some other vertex of S . The cardinality of a minimum supergeodetic set is called the *supergeodetic number* $sg(G)$ of G . Note that, in general, this parameter does not agree with the geodetic number of the graph. For instance, the graph of Figure 8 has geodetic number 3, and $S = \{u, v, w\}$ is the unique minimum geodetic set, whereas the supergeodetic number of G is 4, and $S' = \{u, v, w, x\}$ is a minimum supergeodetic set with distinguished vertex u .

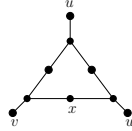


Figure 8: A graph with geodetic number 3 and supergeodetic number 4.

We study some cases in which both parameters, g and sg , agree, and also we give examples with any suitable geodetic and supergeodetic numbers. So both parameters agree when extreme values are reached and, in contrast, the difference between geodetic and supergeodetic numbers can be as high as we want.

- Theorem 1** 1. If G is a graph with n vertices, then $sg(G) = 2$ (respectively $n, n - 1$) if and only if $g(G) = 2$ (respectively $n, n - 1$).
2. For positive integers a and b with $3 \leq a \leq b$, there exists a graph G such that $g(G) = a$ and $sg(G) = b$.

It is also natural to relate geodecity parameters with the order and the diameter of the graph as we show in the next result.

Theorem 2 Given k, n and d positive integers such that $k \geq 4, n \geq 6, 2 \leq d \leq n - 4$ and $k < n - d + 1$, then there exists a graph G having n vertices, diameter d and $g(G) < sg(G) = k$.

Regarding to the relationship between supergeodetic number and Steiner number of a graph the results are different. Recall that the *Steiner number* [4] of G , $st(G)$ is the cardinality of the minimum set $S \subset V(G)$ such that any vertex $v \in V(G)$ is in some Steiner tree of S , and it is well known that $g(G) \leq st(G)$. However, in general, there exists no relationship between Steiner and supergeodetic numbers.

Since both geodetic and supergeodetic sets must always contain the extreme vertices of the graph, it is natural to wonder about graphs whose extreme vertex set would be a supergeodetic set. The answer to that question is that extreme vertices are supergeodetic set in trees but not in other close graph classes. So it is interesting to find a universal supergeodetic set, and the boundary vertex set [3] is. Given a graph $G = (V, E)$ and two vertices $u, v \in V$, the vertex v is a *boundary vertex* of u if no other neighbor of v is further from u than v . A vertex v is a boundary vertex of G if it is a boundary vertex of some vertex $u \in V$. The *boundary set* $\partial(G)$ of G is the set of all the boundary vertices and it is a supergeodetic set in any graph G .

Finally we consider the class of split graphs. The geodetic number of any split graphs was obtained in [2], and authors also described minimum geodetic sets in these graphs. We study the supergeodetic number in this graph class. In some cases minimum geodetic sets are also supergeodetic, and in other ones they are not, but can be enlarged to become minimum supergeodetic sets.

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Radio Labelings of Distance Graphs

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EXTENDED ABSTRACT

A *radio k -labeling* of a connected graph G is an assignment f of non negative integers to the vertices of G such that

$$|f(x) - f(y)| \geq k + 1 - d(x, y),$$

for any two distinct vertices x and y , where $d(x, y)$ is the distance between x and y in G . The *radio k -labeling number* $\text{rl}_k(G)$ of G is the minimum of $\max_{x, y \in V(G)} |f(x) - f(y)|$ over all radio k -labelings f of G .

The study of radio k -labelings was initiated by Chartrand et al. [1], motivated by radio channel assignment problems with interference constraints.

Except for paths [1, 3] and cycles [5], radio k -labelings have been investigated mainly for fixed values of k . This problem generalizes both the classical proper vertex-colouring problem (when $k = 1$) and the well studied $L(2, 1)$ -labeling problem (when $k = 2$). The other values of k considered were when k is close to the diameter of the graph. The interested reader is referred to surveys [2, 7] and recent papers [6, 8] for complementary results.

For a set of positive integers $\{d_1, d_2, \dots, d_t\}$, the (infinite) distance graph $D(d_1, d_2, \dots, d_t)$ has the set \mathbb{Z} of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $|i - j| = d_\ell$, for some ℓ .

Concerning radio k -labelings of distance graphs, the only known results are for $k = 2$ and mainly for 4-regular distance graphs [4, 9]. Moreover, for the path P_n of order n (a finite subgraph of $D(1) = P_\infty$), the following bounds were proved in [1, 3]: for any $n > 3$ and any $1 \leq k \leq n - 3$,

$$\frac{k^2 + 4}{2} \leq \text{rl}_k(P_n) \leq \frac{k^2 + 2k}{2}, \text{ if } k \text{ is even,}$$

$$\frac{k^2 + 1}{2} \leq \text{rl}_k(P_n) \leq \frac{k^2 + 2k - 1}{2} \text{ if } k \text{ is odd;}$$

and it was conjectured in [3] that the upper bound is the exact value of the radio k -labeling number when the length of the path is large enough.

We prove the following results :

$$\frac{t}{2}k^2 + \frac{1}{2} \leq \text{rl}_k(D(1, 2, \dots, t)) \leq \begin{cases} \frac{t}{2}k^2 + \frac{t}{2}k, & \text{when } k \text{ is odd,} \\ \frac{t}{2}k^2 + k, & \text{when } k \text{ is even.} \end{cases}$$

$$\begin{aligned} \frac{t}{2}k^2 - P_2(t)k + P_3(t) &\leq \text{rl}_k(D(1, t)) \leq \frac{t}{2}k^2, & \text{for } t \geq 3 \text{ and odd } k, \\ \frac{t}{2}k^2 - Q_2(t)k + Q_3(t) &\leq \text{rl}_k(D(t - 1, t)) \leq \frac{t}{2}k^2 + k - \frac{t+2}{2}, & \text{for } t \geq 3 \text{ and odd } k. \end{aligned}$$

where $P_i(t)$ and $Q_i(t)$ denote polynomials of variable t of degree i .

For each upper bound, we have found a corresponding coloring sequence with the desired number of labels while lower bounds were obtained by bounding the upper traceable number of the distance graphs by a function of the same parameter on the infinite path.

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Rainbow Connection in Hypergraphs

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EXTENDED ABSTRACT

The subject of *rainbow connection* in graphs was introduced by Chartrand et al [2] in 2008. Given a simple, finite, connected graph G , the *rainbow connection number* of G , denoted by $rc(G)$, is defined as follows. An edge-coloured path is *rainbow* if its edges have distinct colours. Then, $rc(G)$ is the minimum number of colours required to colour the edges of G such that any two vertices are connected by a rainbow path.

Chartrand et al [2] studied the function $rc(G)$ for many basic graphs G , including cycles, and complete bipartite and multipartite graphs. They also introduced a related function, the *strong rainbow connection number* $src(G)$, which is the minimum number of colours required to colour the edges of G such that any two vertices are connected by a rainbow geodesic. Since then, the rainbow connection subject has attracted considerable interest. These include the study of $rc(G)$ in relation to the minimum degree, or forbidden subgraphs, or diameter of G , or when G is regular. Further related functions have also been introduced, such as the *rainbow vertex connection number* $rvc(G)$, which is the version with vertex-colourings, and the *rainbow k -connection number* $rc_k(G)$, which is the version involving edge-colourings where any two vertices of G are connected by k internally vertex-disjoint rainbow paths (here, G is a k -connected graph). See for example, Caro et al [1], Chartrand et al [3], Fujita et al [4], and Krivelevich and Yuster [5]. Recently, Li and Sun published a survey [6] and a book [7] on the current status of rainbow connection.

Here, we introduce the rainbow connection notion for hypergraphs. We only consider hypergraphs which are finite, undirected and without multiple edges. An obstacle in the hypergraphs setting is that the notion of a path is not obvious. To clarify this, we will actually consider two types of paths. Firstly, for $\ell \geq 1$, a *Berge path*, or simply a *path*, is a hypergraph consisting of a sequence $v_1, e_1, v_2, e_2, \dots, v_\ell, e_\ell, v_{\ell+1}$, where $v_1, \dots, v_{\ell+1}$ are distinct vertices, e_1, \dots, e_ℓ are distinct edges, and $v_i, v_{i+1} \in e_i$ for every $1 \leq i \leq \ell$. Secondly, for $\ell \geq 1$ and $1 \leq s < r$, an (r, s) -*path* is an r -uniform interval hypergraph where the intervals can be linearly ordered so that consecutive intervals intersect in s vertices.

An edge-coloured path or (r, s) -path is *rainbow* if its edges have distinct colours. For a connected hypergraph \mathcal{H} , the *rainbow connection number* of \mathcal{H} , denoted by $rc(\mathcal{H})$, is the minimum number of colours required to colour the edges of \mathcal{H} such that any two vertices are connected by a rainbow path. Similarly for $1 \leq s < r$, let $\mathcal{F}_{r,s}$ be the family of the hypergraphs where any two vertices are connected by an (r, s) -path. For $\mathcal{H} \in \mathcal{F}_{r,s}$, the (r, s) -*rainbow connection number* of \mathcal{H} , denoted by $rc(\mathcal{H}, r, s)$, is the minimum number of colours required to colour the edges of \mathcal{H} such that any two vertices are connected by a rainbow (r, s) -path.

Hence, we have two generalisations of the rainbow connection number from graphs to hypergraphs. There are good reasons as to why we consider both generalisations. We consider the version with Berge paths because this covers the situation for a larger class of hypergraphs, namely, all connected hypergraphs, rather than just the class $\mathcal{F}_{r,s}$ for the (r, s) -paths version. On the other hand, for many hypergraphs, the version with the (r, s) -paths is more interesting than the one with the Berge paths.

We shall present some results about the functions $rc(\mathcal{H})$ and $rc(\mathcal{H}, r, s)$. Firstly, note that we have $rc(\mathcal{H}, r, s) \geq rc(\mathcal{H})$ for all $1 \leq s < r$ and $\mathcal{H} \in \mathcal{F}_{r,s}$.

Proposition 1 *Let $r \geq 3$ and $1 \leq s \neq s' < r$. Then, $rc(\mathcal{H}, r, s) - rc(\mathcal{H})$ can be arbitrarily large for $\mathcal{H} \in \mathcal{F}_{r,s}$, and $rc(\mathcal{H}, r, s) - rc(\mathcal{H}, r, s')$ can be arbitrarily large for $\mathcal{H} \in \mathcal{F}_{r,s} \cap \mathcal{F}_{r,s'}$.*

Hence, Proposition 1 says that the functions $rc(\mathcal{H})$, $rc(\mathcal{H}, r, s)$ and $rc(\mathcal{H}, r, s')$ are “separated” from one another, and in the latter case, we cannot bound one of $rc(\mathcal{H}, r, s)$ and $rc(\mathcal{H}, r, s')$ in terms of the other.

Next, note that we have $rc(\mathcal{H}) \leq e(\mathcal{H})$. Our next result characterises those hypergraphs \mathcal{H} where $rc(\mathcal{H}) = e(\mathcal{H})$. A hypergraph \mathcal{H} with $e(\mathcal{H}) \geq 1$ is *minimally connected* if \mathcal{H} is connected, and $(V(\mathcal{H}), E(\mathcal{H}) \setminus e)$ is disconnected for every $e \in E(\mathcal{H})$.

Theorem 2 *Let \mathcal{H} be a connected hypergraph with $e(\mathcal{H}) \geq 1$. Then, $rc(\mathcal{H}) = e(\mathcal{H})$ if and only if \mathcal{H} is minimally connected.*

We remark that in the simple graphs setting, a graph is minimally connected if and only if it is a tree. However, this fact does not carry over to hypergraphs. That is, minimally connected hypergraphs are generally not hypertrees, and vice versa. Hence, we do not necessarily have $rc(\mathcal{H}) = e(\mathcal{H})$ if \mathcal{H} is a hypertree.

Finally, we consider $rc(\mathcal{H}, r, s)$ for certain hypergraphs \mathcal{H} . For $2 \leq r < n$, let C_n^r denote the r -uniform cycle on n vertices, where every r consecutive vertices form an edge.

Theorem 3 *Let $1 \leq s < r < n$. Then for sufficiently large n , we have $rc(C_n^r, r, s) = \frac{n}{2(r-s)} + c_{r,s}$, where $c_{r,s}$ depends on r and s .*

For $t \geq r \geq 3$ and $1 \leq n_1 \leq \dots \leq n_t$, let $\mathcal{K}_{n_1, \dots, n_t}^r$ denote the r -uniform complete multipartite hypergraph with class sizes n_1, \dots, n_t .

Theorem 4 *Let $t \geq r \geq 3$, $s \leq r - 2$ and $1 \leq n_1 \leq \dots \leq n_t$ be such that $n_t = 1$ or $n_{2(t-r)+s+1} \geq 2$. Then,*

$$rc(\mathcal{K}_{n_1, \dots, n_t}^r, r, s) = \begin{cases} 1 & \text{if } n_t = 1, \\ 2 & \text{if } n_t \geq 2. \end{cases}$$

Theorem 5 *Let $t \geq r \geq 3$, $1 \leq n_1 \leq \dots \leq n_t$, $n = n_t$ and $b = \sum_{S \in [t-1]^{(r-1)}} \prod_{i \in S} n_i$, where $[t-1]^{(r-1)}$ denotes the family of subsets of $\{1, \dots, t-1\}$ with size $r-1$. Then,*

$$rc(\mathcal{K}_{n_1, \dots, n_t}^r, r, r-1) = \begin{cases} 1 & \text{if } n_t = 1, \\ \lceil \sqrt[b]{n} \rceil & \text{if } t = r \text{ and } n_1 = 1, \\ \min(\lceil \sqrt[b]{n} \rceil, r+2) & \text{if } t = r \text{ and } n_1 \geq 2, \\ \min(\lceil \sqrt[b]{n} \rceil, 3) & \text{if } t > r. \end{cases}$$

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Minmax Degree of planar graphs without 4-cycles and i -cycles

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EXTENDED ABSTRACT

The *minmax degree* of a graph G , denoted by $M^*(G)$, is the greatest integer k such that each edge of G is incident to a vertex with degree at least k .

In this talk we will present some results on the minmax degree of the family of planar graphs without cycles of length 4 and minimum degree at least 2 (in the following we will denote by $\delta(G)$ the minimum degree of the graph G). This graph invariant has some interesting applications for the edge decomposition problem, from which we can derive some corollaries on the marking game.

If the edge set of the graph G can be partitioned into two subsets F and D such that $G[F]$ the graph induced by F is a forest and $G[D]$ the graph induced by D is a subgraph with maximum degree at most d , then G is said $(1, d)$ -coverable. Minmax degree and edge decomposition are related by the following theorem due to He et al.:

Theorem 1 (He et al. [1]) *Let G be a graph. If every subgraph H of G has $M^*(H) \leq k$ or $\delta(H) \leq 1$, then G is $(1, k)$ -coverable.*

The *marking game* on a graph G , introduced by Zhu in [2], is a two-player non-cooperative game: Alice and Bob take turns selecting an unselected vertex of G , establishing a linear order over the vertices of G . The *back degree* of a vertex v of G is the number of neighbors of v selected before v , and the *score* of the game is equal to $1 + b$, where b is the maximum back degree among the vertices of G . Alice has to minimize the score and Bob to maximize it. The game coloring number $col_g(G)$ of G is the greatest score of a game on G when Alice uses her optimal strategy.

The following lemma gives the link between edge decomposition and marking game.

Lemma 2 (He et al. [1], Zhu [2], Faigle et al. [3]) *Let G be a $(1, k)$ -coverable graph. We have $col_g(G) \leq k + 4$.*

The minmax degree of a planar graph with $\delta(G) \geq 2$ cannot be bounded, since $K_{2,n}$ has a minmax degree equal to n . However, we can characterize the minmax degree of a planar graph G using its *2-stackability* $S_2(G)$, that is the least integer k such that each couple of vertices has at most k common neighbors of degree 2:

Theorem 3 (C., M., R. [4]) *Let G be a planar graph with $\delta(G) \geq 2$. We have $M^*(G) \leq 5(S_2(G) + 2)$.*

Tight bounds are known for planar graphs with minimum degree at least 3 (see [5, 6, 7]), or for planar graphs with $\delta(G) \geq 2$ and girth at least 5 (see [1]). Consider now the family of planar graphs without cycles of length 4. Borodin et al. proved the following theorem:

Theorem 4 (Borodin et al. [8, 9]) *Let G be a planar graph with $\delta(G) \geq 2$.*

- *If G does not have 4-cycles, then $M^*(G) \leq 7$.*
- *If G does not have 4- and 5-cycles, then $M^*(G) \leq 5$.*
- *If G does not have cycles of length 4 to 7, then $M^*(G) \leq 4$.*
- *If G does not have intersecting triangles and cycles of length 4 to 9, then $M^*(G) \leq 3$.*

In this talk we will present the following result:

Theorem 5 *Let G be a planar graph with $\delta(G) \geq 2$. If G does not have 4- and i -cycles, for $i \in \{6, 7, 8, 9\}$, then $M^*(G) \leq 5$.*

We will give the proof of the case $i = 6$ and a sketch of proof for the cases 7 and 9.

Moreover we will exhibit examples of planar graphs G with $\delta(G) \geq 2$ and without 4- and i -cycles for $i \in \{5, 7, 9\}$ and with minmax degree 5, proving that for the case $i \in \{5, 7, 9\}$ the upper bound cannot be improved.

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Graphs with cyclomatic number three having panconnected square

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EXTENDED ABSTRACT

All graphs considered in this paper are simple. That is, they have neither loops nor multiple edges. Let G be a graph. The *square* of G , denoted G^2 , is the graph obtained from G by adding edges joining those pairs of vertices whose distance from each other in G is two. A graph is *panconnected* if, between any pair of distinct vertices, it contains a path of each length at least the distance between the two vertices.

A cut edge xy of G is termed an *internal cut edge* if both the degrees of x and y in G are at least 2.

Theorem 1 (Chia, Ong, Tan '2009 [2]) *Let G be a graph. If G^2 is panconnected, then G contains no internal cut edge.*

If G is a connected graph, then the number $c(G) = |E(G)| - |V(G)| + 1$ is called the *cyclomatic number* of G . The problem of characterizing graphs with given cyclomatic number having panconnected square was first posed and studied in [2] where the authors showed that if $c(G) = 0$, then G^2 is panconnected if and only if $G \cong K_{1,n}$, the tree with all vertices but one of degree 1. This is in fact a direct consequence of Theorem 1 since $c(G) = 0$ if and only if G is a tree. The authors in [2] went on to characterize graphs G with $c(G) = 1$ and G^2 is panconnected.

An *SF graph* is a graph consisting of a cycle $x_1x_2 \dots x_mx_1$ together with a set of independent vertices A_i (which may be empty) each joining to the vertex $x_i, i = 1, 2, \dots, m$.

Theorem 2 (Chia, Ong, Tan '2009 [2]) *Let G be a unicyclic graph. Then G^2 is panconnected if and only if G is an SF graph with some degree-2 vertex.*

The case $c(G) = 2$ and G^2 is panconnected was completely settled in [1].

In the present paper, we investigate the case $c(G) = 3$ and G^2 is panconnected. First, we show that a graph with this property must belong to one of the 8 families of graphs denoted $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_8$. We then define three large families of graphs which are generalization of the families $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_5 . Necessary and sufficient conditions for these three large families of graphs to have panconnected square are determined. Hence the classification of graphs in $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_5 having panconnected square follows as a byproduct of these results.

In proving these results, we make use of a well-known theorem of Fleischner [3] which states that if G is a connected graph, then G^2 is panconnected if and only if G^2 is Hamilton-connected.

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Excluding paths and antipaths

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EXTENDED ABSTRACT

The Erdős-Hajnal conjecture states that for every graph H , there exists a constant $\delta(H) > 0$, such that if a graph G has no induced subgraph isomorphic to H , then G contains a clique or a stable set of size at least $|V(G)|^{\delta(H)}$. This conjecture is still open. We consider a variant of the conjecture, where instead of excluding H as an induced subgraph, both H and H^c are excluded. We prove this modified conjecture for the case when H is the five-edge path. Our second main result is an asymmetric version of this: we prove that for every graph G such that G contains no induced six-edge path, and G^c contains no induced four-edge path, G contains a polynomial-size clique or stable set.

Fixed point properties for Lexicographic Breadth First Search on cocomparability graphs

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EXTENDED ABSTRACT

Since the 1950s, search, a mechanism for visiting the vertices and edges of a graph, has been a fundamental technique in the design of graph algorithms. Recently some algorithms were proposed made up with a series of consecutive graph searches [4], these algorithms are also called **multisweep algorithms**. Here we study a particular case of multisweep algorithms, for which it exists some rule for the composition of searches. To do so we focus on the ordering of the vertices as the result of a graph search, now we can compose graph searches in a natural way, since graph searches can be viewed as operators acting on graphs and producing total orderings of the vertices.

Therefore we can denote by $M(G, x_0)$ the order of the vertices obtained by applying a graph search M on G starting from the vertex x_0 . An easy rule to compose graph search is just to start a new search from the last vertex visited by the previous search. This idea has been successfully used with Breadth First Search for efficient diameter evaluation and computation in graphs [8, 5]. Another interesting compositional rule, namely the $+$ **Rule** was already used in some graph algorithms, mainly based on Lexicographic Breadth First Search (LBFS for short) [1].

Definition: Let M be a graph search and σ an ordering of the vertices of G , $M^+(G, \sigma)$ be the ordering of the vertices obtained by applying M on G starting from the vertex $\sigma(1)$ and tie-breaking using σ in decreasing order.

Using this $+$ Rule, we can now consider the following process:

Step 0: $\sigma = M(G, x_0)$

Step 1: $M^1(G, \sigma) = M^+(G, \sigma)$

Step 2: $M^2(G, \sigma) = M^+(G, M^+(G, \sigma))$

...

Step i : $M^i(G, \sigma) = M^+(G, M^{i-1}(G, \sigma))$

...

This leads to very natural problems : what are the properties of these orderings ? What is the behavior of this infinite sequence of orderings ? Using the $+$ rule, the sequence cannot have a fixed point, but since the graphs we consider are finite this sequence is necessarily periodical. Could this orbit be of size 2, in other words the question is :

Question: For which search M and graph G does there exist fixed points (i.e. some index i for which $M^i(G, \sigma)$ and $M^{i-1}(G, \sigma)$ are two reverse orderings or $M^{i+1}(G, \sigma) = M^{i-1}(G, \sigma)$)?

In this case, the process will infinitely alternate between the orderings σ and τ , dual orderings via the $+$ rule and using an abuse of language we can say that the above computational process has a quasi-fixed point.

The $+$ Rule forces to keep the ordering of the previous sweep in case of tie-break and therefore using a series of searches one can incrementally discover some graph structure and uses this paradigm to design recognition algorithms for special classes of graphs. This yields simple algorithms which are very easy to program, however their proof could be difficult since we do not yet have good tools to analyze such algorithms. Our general aim is to find general tools for proving fixed point properties of such algorithmic schemes.

In particular using this idea, Corneil and Köhler [2] proved that a series of consecutive LBFS applied on an interval graph produces a linear ordering of the vertices which characterizes it (from this ordering which also produces a linear order of the maximal cliques of

the graph, the interval representation can be easily obtained). In fact Corneil and Köhler proved that $O(n)$ consecutive LBFS are enough to produce an interval representation.

A similar property has been used in a recent linear time interval graph recognition algorithms [4]. To achieve linearity this algorithm uses five consecutive LBFS, and terminates using a modified LBFS (which uses a different tie-break based on the two previous ones).

Here we study the generalization of Corneil and Köhler's result to cocomparability graphs (graph whose complement can be transitively oriented).

Definition: For a total ordering τ of the set of vertices, an umbrella is a triple of vertices $a, b, c \in X$ such that: $a <_\tau b <_\tau c$ and $ac \in E$ and $ab, bc \notin E$.

A co-comparability (co-comp for short) ordering is an umbrella-free total ordering of the vertices of G . The following result was first noticed in [7].

Proposition: G is a co-comparability graph iff it admits a co-comp ordering.

In fact we prove that a series of $O(n)$ consecutive LBFS applied on a cocomparability graph produces a cocomp ordering. In order to prove this result we first establish structural properties of the maximal clique lattice of the cocomparability graph, and study the trace of LBFS on this lattice.

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The atomic decomposition of strongly connected graphs

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EXTENDED ABSTRACT

We introduce a canonical decomposition of strongly connected graphs that we call the *atomic decomposition*. We can construct it in linear time from the canonical decomposition in 3-connected components of the considered graph. With it, we study *Gauss words*: these words are associated with self-intersecting closed curves in the plane. We characterize, up to homeomorphism, all such curves with a given Gauss word. We extend this characterization to tuples of words, associated with tuples of closed curves in the plane.

Knuth has introduced a similar but different decomposition of the same graphs that is not canonical (it is not uniquely associated with a graph), but its indecomposable elements are the vertices. We will compare it to the atomic decomposition.

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(Nearly-)Tight Bounds on the Linearity and Contiguity of Cographs

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EXTENDED ABSTRACT

Introduction. Linearity and contiguity are graph parameters introduced to obtain efficient codings of neighborhoods in graphs, by decomposing each neighborhood as a union of p intervals chosen in one or several orders on the vertices [1]. Indeed, storing an order of the vertices as well as a pair of pointers for each of the p intervals of this order (one pointer for the beginning of the interval and one for the end), with fixed p , allows to store the graph in $O(n)$ space (instead of $O(n + m)$ with adjacency lists) and access the neighborhood of any vertex v in $O(d)$ time (instead of $O(n)$ with adjacency matrices), where d is the degree of v .

More formally, a *closed p -interval-model* of a graph $G = (V, E)$ is a linear order σ on V such that $\forall v \in V, \exists (I_1, \dots, I_p) \in (2^V)^p$ such that $\forall i \in \{1, \dots, p\}, I_i$ is an interval of σ and $N[v] = \bigcup_{1 \leq i \leq p} I_i$. The *closed contiguity* of G , denoted by $\text{cont}(G)$, is the minimum integer p such that there exists a closed p -interval-model of G . A *closed p -line-model* of a graph $G = (V, E)$ is a tuple $(\sigma_1, \dots, \sigma_p)$ of linear orders on V such that $\forall v \in V, \exists (I_1, \dots, I_p) \in (2^V)^p$ such that $\forall i \in \{1, \dots, p\}, I_i$ is an interval of σ_i and $N[v] = \bigcup_{1 \leq i \leq p} I_i$. The *closed linearity* of G , denoted by $\text{lin}(G)$, is the minimum p such that there exists a closed p -line-model of G .

Not much is known about these parameters, which cannot be bounded by a constant even in very restricted graph classes, like interval or permutation graphs [1]. We focus here on the contiguity and linearity of cographs (graphs without induced P_4 subgraphs), whose very constrained structure can be represented by their *cotree*, a rooted tree with two kinds of nodes labeled by P and S , giving a tight upper bound for the asymptotic contiguity of cographs and an upper bound for their linearity. To this aim, we first establish a min-max theorem on the link between the rank of rooted trees and their decompositions into paths.

A min-max theorem on the rank of a tree. The *rank* [2, 3] of a tree T is the maximal height of a complete binary tree obtained from T by edge contractions, that is $\text{rank}(T) = \max\{h(T') \mid T' \text{ complete binary tree, minor of } T\}$.

A *path partition* of a tree T is a partition $\{P_1, \dots, P_k\}$ of $V(T)$ such that for any i , the subgraph $T[P_i]$ of T induced by P_i is a path, as shown in Figure 9(a). The *partition tree* of a path partition \mathcal{P} , denoted by $T_p(\mathcal{P})$ and illustrated in Figure 9(b), is the tree whose nodes are P_i 's and where the node of $T_p(\mathcal{P})$ corresponding to P_i is the parent of the node corresponding to P_j iff some node of P_i is the parent in T of the root of P_j . The height of a path partition \mathcal{P} of a tree T , denoted by $h(\mathcal{P})$, is the height $h(T_p(\mathcal{P}))$ of its partition tree. The *path-height* of T is the minimal height of a path partition of T , that is $ph(T) = \min\{h(\mathcal{P}) \mid \mathcal{P} \text{ path partition of } T\}$.

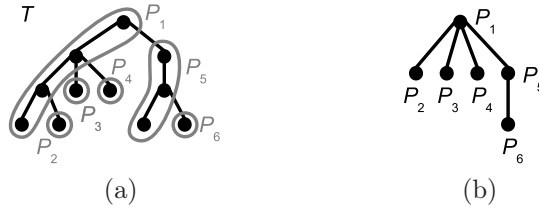


Figure 9: A tree T and a path partition $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ of T (a), as well as the partition tree of \mathcal{P} (b).

Lemma 1 For a rooted complete binary tree T , $\text{rank}(T) = \text{ph}(T) = h(T)$.

Theorem 2 For any rooted tree T , we have $\text{rank}(T) = \text{ph}(T)$.

Upper bounds for contiguity and linearity of cographs. We now combine the results of the previous section with a decomposition of the cotree of the input cograph into paths, in order to obtain a constructive proof that the contiguity of any cograph is at most $O(\log n)$. This decomposition is obtained recursively, using a root-path decomposition of the cotree, thanks to the Caterpillar Composition Lemma below.

A *root-path decomposition* (see Fig. 10) of a rooted tree T is a set $\{T_1, \dots, T_p\}$ of disjoint subtrees of T , with $p \geq 2$, such that every leaf of T belongs to some T_i , with $i \in [1..p]$, and the sets of parents in T of the roots of T_i 's is a path containing the root of T .

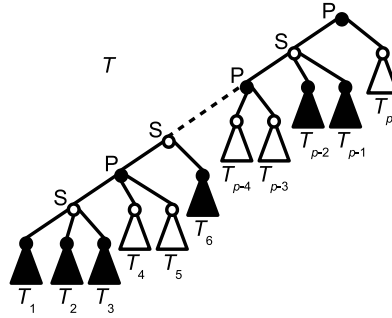


Figure 10: The root-path decomposition $\{T_1, \dots, T_p\}$ of a rooted tree T .

Lemma 3 (Caterpillar Composition Lemma) Given a cograph $G = (V, E)$ and a root-path decomposition $\{T_i\}_{1 \leq i \leq p}$ of its cotree, where X_i is the set of leaves of T_i , $\text{cont}(G) \leq 2 + \max_{i \in [1..p]} \text{cont}(G[X_i])$.

Lemma 4 Given a rooted tree T such that $\text{rank}(T) = k \geq 1$, there exists a root-path decomposition $\{T_1, \dots, T_p\}$ of T such that for each $i \in [1..p]$, $\text{rank}(T_i) \leq k - 1$.

Lemma 5 Let G be a cograph and T its cotree. We have $\text{cont}(G) \leq 2\text{rank}(T) + 1$.

Theorem 6 The closed contiguity of a cograph is at most logarithmic in its number of vertices, or more formally, if $G = (V, E)$ is a cograph, then $\text{cont}(G) \leq 2\log_2 |V| + 1$.

Lower bounds for contiguity and linearity of cographs. Finally, we focus on cographs whose cotrees are complete binary trees, and obtain a tight lower bound for their asymptotic contiguity as well as a lower bound for their asymptotic linearity.

Theorem 7 Let G be a cograph whose cotree is a complete binary tree. Then, $\text{cont}(G) = \Omega(\log n)$ and $\text{lin}(G) = \Omega(\log n / \log \log n)$.

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On stable matchings in the Gale-Shapley model

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EXTENDED ABSTRACT

The well-known two-sided matching problem has many applications in economics, computer science and other disciplines (see [2], [4], [5]). It was introduced by Gale and Shapley in their famous paper [3] and presented in the simplest setting as a problem of forming marriages between n men and n women (interchangeably, as a college admission problem of size n). The key concept in the Gale-Shapley theory is the concept of stability of a given set of marriages. It can be expressed in the graph theory language as a perfect matching problem in a complete bipartite graph $K_{n,n}$ that satisfies some additional requirements. In this model, each vertex v has assigned a preference list consisting of all the vertices of the opposite bipartition set, whose elements are linearly ordered by \prec_v . A perfect matching N of such a graph is not stable if there are vertices x, y from different bipartition sets, such that $xy \notin N$ and for a, b satisfying $xa \in N$, $yb \in N$ there hold $a \prec_x y$ and $b \prec_y x$. It can be easily proved that for given preferences there exists at least one stable perfect matching (stable set of marriages, see [3]), but counting the exact number of all stable perfect matchings remains an open question till today. Some lower bounds on the maximum number of all stable perfect matchings, in the problem of size n , are known (see [1], [2]), but no non-trivial upper bound is not known yet.

In this work we give some upper bounds on the maximum number of stable matchings in the Gale-Shapley model (Theorems 1, 2). Using some graph terms, we also compute the exact number of such matchings, when the preferences are fixed (Theorem 3). Some sufficient conditions for the uniqueness of a stable matching in the Gale-Shapley model of even size are given (Theorem 4). The result on the existence of exactly two stable matchings in the problem of odd size, satisfying the same conditions, can be found too (Theorem 5).

We start with two finite n -element sets, say, the set of women $W = \{W_1, \dots, W_n\}$ and the set of men $M = \{M_1, \dots, M_n\}$. Each man and each woman linearly orders all the members of the opposite sex. The preferences of the sexes are represented by two tables \mathbf{W} and \mathbf{M} , each of n^2 elements (of size n). The i^{th} row of the table \mathbf{W} shows preferences of W_i and is of the form M_{i_1}, \dots, M_{i_n} , where $\{M_{i_1}, \dots, M_{i_n}\} = M$ (referring to it we write $(W_i : M_{i_1}, \dots, M_{i_n})$). This notation informs us that for $l < m$ the woman W_i prefers M_{i_l} to M_{i_m} . If there are exactly a men, who are worse than M_j for W_i , then notation $(W_i : M_j, a)$ is used. Analogous description for the table \mathbf{M} will be applied. In the case, when we only know, or we are only interested in, some parts of this list, the symbol " $|$ " substitutes possible elements of the list. For instance, $(W_i : |M_{i_1}, M_{i_2}|M_{i_3}|)$ means that W_i prefers M_{i_1} to M_{i_2} ; M_{i_1} to M_{i_3} ; and M_{i_2} to M_{i_3} . Moreover at W_i 's list of preferences there is no element of the set M located between M_{i_1} and M_{i_2} . Similarly, $(W_i : M_{i_1}, M_{i_2}|M_{i_3}|)$ denotes all the assertions implied by the notation $(W_i : |M_{i_1}, M_{i_2}|M_{i_3}|)$ with additional information that M_{i_1} is the first element at W_i 's list of preferences.

By $\sigma(\mathbf{W}, \mathbf{M})$ we denote a number of all stable perfect matchings in a complete bipartite graph with vertex bipartition (W, M) and vertex preferences given by \mathbf{W}, \mathbf{M} .

Theorem 1 *Let $n \geq 2$. If \mathbf{W} and \mathbf{M} are given tables of preferences of size n , then*

$$\begin{aligned} \sigma(\mathbf{W}, \mathbf{M}) &\leq \frac{3n^2 - 4n + 1}{4}(n-2)! && \text{for } n \text{ odd, and} \\ \sigma(\mathbf{W}, \mathbf{M}) &\leq \frac{3n^2 - 2n}{4}(n-2)! && \text{for } n \text{ even,} \end{aligned}$$

which means that in both cases $\sigma(\mathbf{W}, \mathbf{M})$ is not greater than three quarters of all possible matchings.

In what follows $X_{a,b}(\mathbf{W}, \mathbf{M}) = \{(i, j) : (W_i : M_j, a) \text{ and } (M_j : W_i, b)\}$.

Theorem 2 *Let $n \geq 2$ be an even integer. If \mathbf{W} and \mathbf{M} are tables of preferences of size n and there exist $a, b \in \{0, \dots, n-1\}$ such that $a \leq \frac{n}{2} - 1$ and $b \leq \frac{n}{2} - 1$ and $X_{a,b} \neq \emptyset$, then*

$$\sigma(\mathbf{W}, \mathbf{M}) \leq \frac{3n^2 - 4n}{4}(n-2)!.$$

In the theorem below the symbol $\mathcal{F}_s^k(\mathbf{W}, \mathbf{M})$ denotes a set of k -matchings produced by s 2-matchings in a special graph defined by \mathbf{W} and \mathbf{M} .

Theorem 3 *Let $n \geq 2$. If \mathbf{W} and \mathbf{M} are tables of preferences of size n , then*

$$\sigma(\mathbf{W}, \mathbf{M}) = n! + \sum_{s=1}^{n(n-1)} (-1)^s \sum_{k=2}^n (n-k)! |\mathcal{F}_s^k(\mathbf{W}, \mathbf{M})|.$$

We shall say that tables of preferences \mathbf{W}, \mathbf{M} of size n satisfy conditions $(*)$ if the following hold:

1. $(W_1 : M_2, M_1|)$, and
2. $(W_{n-1} : M_{n-2}, M_{n-1}|M_n|M_1|)$, and
3. $(W_i : M_{i-1}, M_i|)$ for $i \in [n] \setminus \{1, n-1\}$, and
4. $(M_1 : W_{n-1}, W_1|)$, and
5. $(M_2 : W_n, W_2, |W_3|W_1|)$, and
6. $(M_n : W_{n-1}, W_n|)$, and
7. $(M_i : W_{i-2}, W_i|)$ for $i \in [n] \setminus \{1, 2, n\}$.

Theorem 4 *Let $n \geq 4$. If \mathbf{W} and \mathbf{M} are tables of preferences of even size n that satisfy the conditions $(*)$, then $\sigma(\mathbf{W}, \mathbf{M}) = \{\sigma_1\}$, where $\sigma_1(i) = i$ for all $i \in [n]$.*

Theorem 5 *Let $n \geq 5$. If \mathbf{W} and \mathbf{M} are tables of preferences of odd size n that satisfy the conditions $(*)$, then $\sigma(\mathbf{W}, \mathbf{M}) = \{\sigma_1, \sigma_2\}$, where*

1. $\sigma_1(i) = i$ for all $i \in [n]$, and
2. $\sigma_2(i) = i$ for all odd $i \in [n-2]$ and $\sigma_2(n) = 2$, $\sigma_2(n-1) = n$, $\sigma_2(n-i) = n-i+2$ for all odd $i \in \{3, \dots, n-2\}$.

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Acyclic edge-coloring of graphs using entropy compression

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EXTENDED ABSTRACT

The Lovász Local Lemma is a powerful tool to obtain existential proofs of upper bounds on coloring parameters in graphs. While the order of magnitude of these bounds is most of the time best possible, the multiplicative constant factor is in general far from the optimal. Using the ideas popularized by Moser and Tardos in their algorithmic proof of the Local Lemma [7], based on entropy compression, some work has recently been done to obtain significant improvements on these constants in the context of non-repetitive coloring of graphs [3, 5]. The aim of this paper is to show that these ideas can be applied to a broad variety of graph coloring problems. To illustrate the general technique, we apply these ideas to acyclic edge-colorings of graphs.

An edge-coloring of a graph G is *acyclic* if it is a proper edge-coloring (adjacent edges have different colors) and every cycle contains at least three colors. A corollary of a general theorem of Alon *et al.* [1] proved using the Lovász Local Lemma is that every graph with maximum degree at most Δ has an acyclic edge-coloring with at most 64Δ colors. Molloy and Reed [6] improved the bound to 16Δ in 1998, and this was recently improved by Ndreca *et al.* [8] to 9.62Δ . Here we improve the bound further to 4Δ . Alon *et al.* [2] conjectured in 2001 that the right bound should be $\Delta + 2$ (only one more than the bound of Vizing for proper edge-coloring).

Our proof is based on the analysis of a very simple algorithm. Order the edges of G as e_1, \dots, e_m , and do the following at each step: take the non-colored edge with smallest index, say e_i , and assign it a random color in $\{1, \dots, 4\Delta\}$ that does not appear on some edge adjacent to e_i . If some bicolored cycle is created, then uncolor e_i and all the other edges on this cycle except two of them. This way, we maintain a partial edge-coloring that is acyclic at each step. We show that this algorithm terminates with non-zero probability, which implies that G has an acyclic edge-coloring with at most 4Δ colors.

In order to analyse this randomised algorithm running on a deterministic instance, we consider it instead as a deterministic algorithm taking a large vector with random entries as input. Take some large integer t , and consider any vector $F \in \{1, \dots, 2\Delta\}^t$. At step i of the algorithm, the i -th entry F_i of F is used to assign a color among the set $\{1, \dots, 4\Delta\}$ to the non-colored edge e_j with smallest index, in such a way that the coloring is proper and there is no bicolored cycle of length four containing e_j (these two requirements lead to a set of at most 2Δ forbidden colors for e_j , so at least 2Δ colors from the set $\{1, \dots, 4\Delta\}$ remain available for e_j).

The key of the analysis is to keep a (compact) record of each step of the algorithm, in such way that at any step i , the record until step i and the partial coloring at step i are enough to deduce all the entries F_j , $j \leq i$. To do so, we define a vector $R = (R_i)_{i \leq t}$ with t entries, entry R_i corresponding to the record at step i . Assume that at step i of the algorithm, the edge e_j was colored and a bicolored cycle was created, say $C = e_{i_1}, \dots, e_{i_{2k}}, e_{i_1}$ with $e_{i_1} = e_j$ and $k \geq 3$. Observe that there are at most Δ^{2k-2} cycles of length $2k$ containing e_j , so we can fix an order on such cycles (say the lexicographic order), as C_1, C_2, \dots, C_s , with $s \leq \Delta^{2k-2}$. In this case we uncolor all the edges of C except e_2 and e_3 , and we set the i -th entry R_i of R to be equal to the pair (k, ℓ) , where $\ell \leq s$ is the index of C among all cycles of length $2k$ containing e_j . If no bicolored cycle is created at step i , R_i is left empty.

Let \mathcal{F}_t be the set of vectors F such that at step t of the algorithm, the graph G has not been completely colored. Let \mathcal{R}_t be the set of records R that can be produced with inputs

from \mathcal{F}_t . Since one record together with one partial coloring determines at most one input vector F , we have:

Lemma 1 $|\mathcal{F}_t| \leq (4\Delta + 1)^m |\mathcal{R}_t|$.

To count the number of possible records $|\mathcal{R}_t|$, we transform any record R into a binary word R° which is a partial Dyck word with all maximal sequences of 1 of even size and strictly greater than 2. Moreover, the preimage of a word R° has size at most Δ^t under this transformation. Using the method described by Flajolet and Sedgewick in [4], we can bound the number of possible words R° and deduce the following result:

Lemma 2 *There is a constant $c > 0$ such that $|\mathcal{R}_t| \leq ct^{-3/2}(2\Delta)^t$.*

Hence, if t is large enough, $|\mathcal{F}_t| < (2\Delta)^t$. As a consequence, there is at least one vector F for which the algorithm terminates, leading to the following theorem.

Theorem 3 *If G has maximum degree Δ , then G has an acyclic edge-coloring with at most 4Δ colors.*

The same method can be used to improve existing bounds when the graph G has larger girth (length of a shortest cycle):

Theorem 4 *Let G be a graph with maximum degree Δ and girth g . Let $a'(G)$ be the minimum of colors in an acyclic edge-coloring of G .*

1. *If $g \geq 7$, $a'(G) \leq 3.74 \Delta$;*
2. *if $g \geq 53$, $a'(G) \leq 3.14 \Delta$;*
3. *if $g \geq 220$, $a'(G) \leq 3.05 \Delta$.*

This method can be also used for star colorings and more generally for graph colorings that can be defined using forbidden configurations of colors of a given list of graphs. From an algorithmic point of view, if one asks for an acyclic edge-coloring with $(4 + \epsilon)\Delta$ colors (for some $\epsilon > 0$) instead of 4Δ colors, the procedure stops in expected polynomial time.

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On automorphism groups of circulant graphs

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EXTENDED ABSTRACT

In paper [1] the recognition and isomorphism problems for circulant graphs were solved in polynomial time. The solution required in particular a deeper study of circulant Schur rings and Cayley schemes including resolving singularities for them and their automorphism groups. This activity was further developed in paper [2]. The theorem formulated below contains several important properties of the automorphism groups of circulant graphs.

Here by a circulant graph we mean any kind of Cayley graph over a finite cyclic group G . For a transitive permutation group Γ on G by a Γ -section we mean a quotient $S = U/L$ where U and L are subgroups of G being blocks of Γ . The restriction Γ^S of Γ to S is by definition the restriction to U/L of the U -stabilizer in Γ . Following H. Wielandt we say that two permutation groups on G are *2-equivalent* if they have the same orbits with respect to their coordinate-wise actions on the set $G \times G$.

Theorem 1 *Let Γ be the automorphism group of a circulant graph. Then*

- (1) *The restriction of Γ to any primitive Γ -section S equals either the symmetric group $\text{Sym}(S)$ or a subgroup of the holomorph $\text{Hol}(S)$ with $|S|$ prime.*
- (2) *Every non-abelian composition factor of Γ is isomorphic to an alternating group.*
- (3) *The group Γ is 2-equivalent to a solvable group containing all translations if and only if every alternating composition factor of Γ is of prime degree.*

It should be remarked that statement (1) can be considered as a natural generalization of the famous Burnside-Schur theorem on primitive permutation groups with a regular cyclic subgroup (see [3]).

In fact Theorem 1 follows from the corresponding theorem on the automorphism groups of circulant S-rings which was proved in [2]. In its turn the latter theorem is a consequence of Theorem 2 below. Here given an S-ring \mathcal{A} over a finite cyclic group G , by an \mathcal{A} -section we mean a quotient $S = U/L$ where U and L are \mathcal{A} -groups, i.e. subgroups of G that are unions of basic sets of \mathcal{A} . The restriction of \mathcal{A} to S is denoted by \mathcal{A}_S . The section S is called *singular* if $\text{rk}(\mathcal{A}_S) = 2$, $|S| > 2$ and \mathcal{A} has certain restrictive structure with respect to the class of projectively equivalent \mathcal{A} -sections that contains S (for details see [2]). The group $\text{Aut}(\mathcal{A})$ equals by definition the automorphism group of the Cayley scheme over G corresponding to \mathcal{A} .

Theorem 2 *Let \mathcal{A} be a circulant S-ring. Then*

- (1) *For any primitive \mathcal{A} -section S one of the following statements holds:*
 - (a) *S is singular and then $\text{Aut}(\mathcal{A})^S = \text{Sym}(S)$,*
 - (b) *the order of S is prime and $\text{Aut}(\mathcal{A})^S \leq \text{Hol}(S)$.*
- (2) *There exists a subgroup Γ of $\text{Aut}(\mathcal{A})$ such that Γ is 2-equivalent to $\text{Aut}(\mathcal{A})$, contains all translations, and for any singular section S of \mathcal{A} we have $\Gamma^S = \text{Hol}(S)$ if the order of S is prime, and $\Gamma^S = \text{Sym}(S)$ otherwise.*

Corollary 3 *If the order of S is composite, then S is singular and $\text{Aut}(\mathcal{A})^S = \text{Sym}(S)$.*

This is a joint work with I. Ponomarenko.

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Constrained Homomorphism Orders

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EXTENDED ABSTRACT

For given graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ a *homomorphism* $f : G \rightarrow H$ is a mapping $f : V_G \rightarrow V_H$ such that $(u, v) \in E_G$ implies $(f(u), f(v)) \in E_H$. If there exists a homomorphism $f : G \rightarrow H$ we write $G \leq H$. It is well known that \leq (seen as a binary relation) induce a quasi-order on the class of all finite graphs. The equivalence classes contain up to isomorphism unique minimal representative, the graph core. The homomorphism order is the partial order on graph cores induced by \leq .

The homomorphism order has been extensively studied, see [1]. It is a non-trivial result that every countable partial order can be found as a suborder of the homomorphism order. The initial result has been proved in even stronger setting of category theory. Subsequently it has been shown that even more restricted classes of graphs and similar structures (in particular the class of oriented paths) admit this universality property [3].

Pair of graphs (G, H) is called a *gap* if $G < H$ and there is no G' with $G < G' < H$. All gaps in the homomorphism order have been characterized. In fact the only gap is (K_1, K_2) and thus the partial order is dense when this pair is removed. The structure of gaps is richer on the homomorphism order of directed graphs and is closely related to the notion of homomorphism dualities.

Several variants of graph homomorphism also have been studied (such as locally constrained homomorphisms, monomorphisms, embeddings, full homomorphisms and surjective homomorphisms). We call those homomorphisms *constrained homomorphisms*. We consider partial orders induced by constrained homomorphisms and ask the same questions — that is to characterize cores, prove or disprove the universality of the partial order, characterize gaps, etc.

Partial order is *future-finite* (or *past-finite*) if every up-set (or down-set) is finite. Partial order is *universal* if it contains every countable partial order as suborder. Similarly partial order is *future-finite-universal* (or *past-finite-universal*) if every future-finite (or past-finite) partial order can be found as a suborder.

Denote by \vec{C}_k the directed graph formed by a cycle on k vertices with edges oriented in one direction. Further denote by DiCycle the class of graphs formed by all cycles \vec{C}_k , $k \geq 3$, and DiCycles the class of graphs formed by disjoint union of finitely many graphs in DiCycle.

We give a new proof of universality of homomorphism order of restricted classes of graphs:

Theorem 1 *The partial order induced on DiCycle by the existence of homomorphism is future-finite-universal. The partial order induced on DiCycles by the existence of homomorphism is universal.*

In this formulation the universality is surprisingly simple to prove (compared to the rather technical proof of universality on the class of oriented paths in [3]). With this new universality argument we can characterize universality of all constrained homomorphisms orders mentioned:

Theorem 2 1. *Partial order implied by the existence of surjective homomorphism on the class of all finite graphs is future-finite and future-finite-universal.*

2. *Partial orders implied by the existence of embedding, monomorphism or full homomorphism on the class of all finite graphs are past-finite and past-finite-universal.*

3. *Partial orders implied by the existence of locally injective, surjective or bijective homomorphism on the class of all finite graphs are universal.*

4. *Partial order implied by the existence of locally injective homomorphisms on the class of all connected graphs is universal.*
5. *Partial orders implied by the existence of locally surjective or locally bijective homomorphisms on the class of all connected graphs are future-finite and future-finite-universal.*

The results hold for directed or undirected graphs with or without loops allowed.

Observations about past-finiteness and future-finiteness of the partial orders make it easy to answer other questions — the partial orders are not dense and dualities exists only in rather simple cases (because either the down-set or up-set is finite).

We extend results of [1] about properties of order induced by locally constrained homomorphisms on connected graphs and give more detailed description of these partial orders.

A partition of the vertex set of a graph G into disjoint classes is called an *equitable partition* if vertices in the same class have the same numbers of neighbors in all classes of the partition. Any equitable partition is characterized by the associated *degree matrix* whose rows and columns are indexed by the blocks of the partition, and the entry in the i -th row and j -th column describes how many neighbors a vertex from the i -th block has in the j -th block. Every finite graph G admits a unique minimal equitable partition. In this case a canonical ordering can be imposed on the blocks, so the corresponding degree matrix, called the *degree refinement matrix*, $\text{drm}(G)$, is also uniquely defined. We put $G \sim H$ if and only if $\text{drm}(G) = \text{drm}(H)$.

With the help of degree refinement matrices we describe gaps:

Theorem 3 *Let H be a connected graph.*

- (a) *There exists connected graph G such that $\text{drm}(G) = \text{drm}(H)$ and (G, H) is a gap in locally injective homomorphism order iff H contains a cycle.*
- (b) *There exists connected graph G , such that $\text{drm}(G) \neq \text{drm}(H)$ and (G, H) is a gap in locally injective homomorphism order iff H has at least one vertex of degree 1.*

And density pairs:

Theorem 4 *Let G and H be connected graphs such that $\text{drm}(G) \neq \text{drm}(H)$, there is locally injective homomorphisms $G \rightarrow H$ and H has no vertices of degree 1. Then:*

- (a) *There exists connected graph F , such that there are locally injective homomorphisms $G \rightarrow F$ and $F \rightarrow H$, $\text{drm}(F) \neq \text{drm}(G)$ and $\text{drm}(F) \neq \text{drm}(H)$. This graph is strictly in between G and H in the locally injective homomorphism order.*
- (b) *When G has no vertices of degree 1 and H has at least one cycle with a vertex of degree greater than 2, then F can be constructed to have no vertices of degree 1 and contain a cycle with a vertex of degree greater than 2.*

While the statements of both results may seem be technical, they generally show that the partial order induced by locally injective homomorphisms can be seen as a combination of: (1) the past-finite and past-finite-universal partial order on acyclic graphs (coinciding with the embeddings), (2) a future-finite and future-finite-universal order within each non-trivial equivalency class of \sim and (3) a universal partial order in between graphs with different degree decomposition matrices (dense when graphs with vertices of degree 1 are excluded).

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Bounding K_4 -minor-free graphs in the homomorphism order

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EXTENDED ABSTRACT

A *homomorphism* of a graph G to a graph H is a mapping f from $V(G)$ to $V(H)$ such that if two vertices are adjacent in G , their images by f are adjacent in H . A class \mathcal{C} of graphs is said to be *bounded* by some graph H if each graph of \mathcal{C} admits a homomorphism to H ; H is called a *bound* for \mathcal{C} . It is of interest to ask for a bound having specific properties (e.g. having specific odd-girth) with smallest possible order. Such questions are studied e.g. in [1].

The projective cube of dimension $2k$, denoted $PC(2k)$, is the graph obtained from the hypercube of dimension $2k + 1$ by identifying each pair of antipodal vertices. $PC(2k)$ has 2^{2k} vertices and odd-girth $2k + 1$ (the odd-girth of a graph is the length of one of its shortest odd cycles). For example, $PC(2)$ is K_4 and $PC(4)$ is the Clebsch graph.

The following question was asked by R. Naserasr in [1]:

Problem 1 *Given two integers $r \geq k$, what is the smallest subgraph of $PC(2k)$ to which every planar graph of odd-girth $2r + 1$ admits a homomorphism to?*

R. Naserasr [1] showed that this question is related to many important theories and captures problems such as edge-colouring, fractional colouring, and circular colouring planar graphs. Motivated by this question, we study the analogous question for the class of series-parallel graphs, i.e. K_4 -minor-free graphs. Let \mathcal{SP}_{2k+1} be the class of series-parallel graphs of odd-girth at least $2k + 1$. We prove:

Theorem 2 *$PC(2k)$ is a bound for \mathcal{SP}_{2k+1} .*

We use the previous theorem to give a reformulation of the question of finding a bound of odd-girth $2k + 1$ of smallest order for \mathcal{SP}_{2k+1} . We present partial answers for the first three cases:

Theorem 3 *The triangle K_3 , the Wagner graph and the graph G_{16} of Figure 11 are bounds for \mathcal{SP}_3 , \mathcal{SP}_5 and \mathcal{SP}_7 , respectively.*

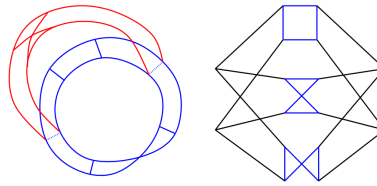


Figure 11: Two drawings of the graph G_{16} that is a bound for \mathcal{SP}_7 .

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γ -cycles in H -edge coloured digraphs.

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EXTENDED ABSTRACT

A digraph D is said to be H -coloured if the arcs of D are coloured with the vertices of H . We will denote by $c(x, y)$ the color of the arc $(x, y) \in F(D)$. An H -walk(path) is a directed walk(directed path) $C = (z_0, z_1, \dots, z_t)$ in D such that $(c(z_0, z_1), c(z_1, z_2), \dots, c(z_{t-1}, z_t))$ is a directed walk in H .

A set N of $V(D)$ is said to be an H -kernel by paths(walks) in D if it satisfies the following two conditions: 1) for every two different vertices u, v in N does not exist an H -path(walk) between them and; 2) for every vertex x in $V(D) - N$ there exists a vertex y in N such that there is an H -path in D from x to y .

A digraph D is a strongly transitive digraph if for any vertices $u, v, w \in V(D)$ (possibly $u = w$) such that $\{(u, v), (v, w)\} \subseteq A(D)$ implies $(u, w) \in A(D)$.

Let H be a strongly transitive digraph (possibly with loops) and D a digraph H -coloured. Let D_1 and D_2 be spanning subdigraphs of D . We will say that $P = \{D_1, D_2\}$ is an H -separation of D if:

1. $F(D_1) \cap F(D_2) = \emptyset$, $F(D_1) \cup F(D_2) = F(D)$
2. Every H -path of D is contained in D_1 or it is contained in D_2 .

In this talk we will show that if H is a strongly transitive digraph and D is a digraph H -coloured, $P = \{D_1, D_2\}$ an H -separation of D such that:

1. Every cycle of D that is contained in D_i is an H -cycle for $i = 1, 2$.
2. D does not contain a (D_1, D, D_2) H -subdivision of C_3
3. If (u, z, w, x_0) is a (D_1, D, D_2) H -subdivision of P_3 then there exist an H -path between u and x_0 .

Then D has a H -kernel by paths.

A γ -cycle in D is a sequence of different vertices, $\gamma = (u_0, \dots, u_n, u_0)$ such that for every $i \in \{0, \dots, n\}$: i) there exists a $u_i u_{i+1} - H$ -path and; ii) there is no $u_{i+1} u_i - H$ -path. We are working in the generalization of the previous result to replace H -cycles by $\gamma - H$ -cycles.

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Shrub-depth, m -partite cographs and MSO_1

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EXTENDED ABSTRACT

One of the most famous algorithmic metatheorems is Courcelle's theorem [2], which states that every graph property expressible in MSO_2 logic of graphs can be solved in linear (FPT) time on graphs of bounded tree-width. More precisely, the MSO_2 model-checking problem for a graph G of tree-width $\text{tw}(G)$ and a formula ϕ , i.e. the question whether $G \models \phi$, can be solved in time $\mathcal{O}(|G| \cdot f(\phi, \text{tw}(G)))$. As shown by Frick and Grohe [3], the function f is, unavoidably, non-elementary in the parameter ϕ (unless $\text{P}=\text{NP}$). However, very recently Gajarský and Hliněný [4] showed that there exists a FPT algorithm for MSO_2 model checking for the class of graphs of bounded tree-depth, with elementary dependency on the formula.

We would like to extend this result to a wider class of graphs, towards graphs of bounded clique-width. To do so, we introduce two new graph parameters: *shrub-depth* and *SC-depth*. Both of these parameters are based on the notion of *tree-model*, which can be seen as a minimalistic analogue of graph interpretation into a tree of bounded height. Shrub-depth and SC-depth are then defined in terms of the number of layers (the depth) such a tree-model must have to be able to interpret a given graph.

We show that the classes of the graphs resulting from an MSO_1 graph interpretation in the class of all finite rooted trees of height $\leq d$, with vertices labelled by a finite set of labels, are exactly the classes of graphs of shrub-depth at most d . This result, in combination with [4], leads to an FPT algorithm for MSO_1 model checking for graphs of bounded shrub-depth (SC-depth) with an elementary dependence on the formula.

We also introduce *m -partite cographs*, which are a natural extension of ordinary cographs. We argue that m -partite cographs represent a smooth intermediate transition from the shrub- and SC-depth to the significantly wider and established notions of clique-width [1] and NLC-width [5]. Indeed, we show that all graphs in any class of shrub-depth d are m -partite cographs with a representation of depth $\leq d$, for suitable m . On the other hand, every m -partite cograph has clique-width at most $2m$. Finally, we show that the class of m -partite cographs is well-quasi-ordered by the relation of “is an induced subgraph of”. This is a significant result, which implies that a) testing whether a graph is an m -partite cograph is a FPT problem, and b) deciding any hereditary property (i.e. property closed under taking induced subgraphs) on this class is a FPT problem.

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Complexity of the (s_1, s_2, \dots, s_k) -Coloring Problem

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EXTENDED ABSTRACT

Let G be a graph. A set $X_i \subseteq V(G)$ is an i -packing if for any distinct pair $u, v \in X_i$, $d_G(u, v) > i$ where $d_G(u, v)$ denotes the usual shortest path distance between u and v . For a sequence of positive integers $1 \leq s_1 \leq s_2 \leq \dots \leq s_k$. The (s_1, s_2, \dots, s_k) -coloring problem $((s_1, s_2, \dots, s_k)$ -COL) is to find k s_i -packings in G , X_{s_1}, \dots, X_{s_k} that form a partition of G .

The $(1, 1, \dots, 1)$ -coloring problem corresponds to the k -coloring problem (where k is the number of 1) which is known to be NP-complete for $k \geq 3$. (s_1, s_2, \dots, s_k) -COL generalizes coloring problems with distance constraints like finding the *packing chromatic number* [1] or finding the *distance chromatic number* [2] of a graph.

The packing chromatic number of G is the least integer k such that the vertex set of G can be partitioned into sets X_i , $i = 1, \dots, k$ where each X_i being an i -packing. The d -distance chromatic number [2] of G is the least integer k such that the vertex set of G can be partitioned into sets X_d^i , $i = 1, \dots, k$ where each X_d^i being a d -packing.

The complexity of (s_1, s_2, s_3) -COL have been studied by Goddard et al. [1] in order to find the complexity of the packing chromatic number when $k = 4$. They have proven that $(1, 1, 2)$ -COL is NP-hard and that $(1, 2, 3)$ -COL is polynomial. Moreover, $(2, 2, 2)$ -COL is polynomial (as only paths of any length and cycles of length of multiple 3 are $(2, 2, 2)$ -colorable). The complexity of $(1, 2, 2)$ -COL was an open question brought by Goddard et al. The next theorem is an answer to this open question.

Theorem 1 $(1, 2, 2)$ -COL is NP-hard.

The following result is signifiant, as it classifies the complexity of all instance of (s_1, s_2, s_3) -COL. However only the hardness of several problems near the bound between the two complexity classes has a significative difficulty (like $(1, 2, 2) - COL$).

Theorem 2 (k_1, k_2, k_3) -COL is polynomial if $k_1 \geq 2$ or if $k_1 = 1$ and $k_2 \geq 3$ or if $k_1 = 1$, $k_2 = 2$ and $k_3 \geq 3$. Otherwise (k_1, k_2, k_3) -COL is NP-hard.

Our goal was to classify the complexity of all instance of (s_1, \dots, s_n) -COL. Some problem has not been classified, however the two following theorems classify numerous problems of (s_1, \dots, s_n) -coloring.

Theorem 3 Let n be a positive integer and $a_k = 2^k - 1$. $(a_1, a_2, \dots, a_n, 3 \times 2^{n-1} - 1, 3 \times 2^{n-1} - 1)$ -COL and $(a_1, a_2, \dots, a_n, a_n, 3 \times 2^{n-1} - 1)$ -COL are NP-hard.

Theorem 4 Let n be an positive integer and a_k be a sequence of n positive integer such that $1 \leq a_1 \leq a_2 \leq \dots \leq a_n$. If $\sum_{i=1}^n \frac{1}{a_i+1} < 1$ then (a_1, a_2, \dots, a_n) -COL is polynomial. Moreover, let odd: $\mathbb{N}^* \rightarrow \mathbb{N}^*$ be the map which associates $n+1$ if n is even and n if n is odd. If $a_1 = 1$ then if $\sum_{i=1}^n \frac{1}{\text{odd}(a_i)+1} < 1$ then (a_1, a_2, \dots, a_n) -COL is polynomial.

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Connectivity Labeling Scheme for Graphs on Surface

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EXTENDED ABSTRACT

We consider the problem of assigning each vertex of a graph G a (short) label such that, for every subset X of vertices of G and every vertex u of $G \setminus X$, the connected component of u in $G \setminus X$ can be (quickly) identified from the labels of u and those of all the vertices of X .

For planar graphs with n vertices we present a new labeling scheme with $O(\log n)$ -bit labels and $O(\log \log n)$ query time after an almost linear time preprocessing of X . The scheme is significantly simpler than previous solution due to Courcelle et al. [3] and the query time is exponentially faster. Our solution generalized to Euler genus- g graphs, and more generally to graphs having some constrained k -page embedding.

Introduction

A natural question in graph theory is to ask whether two vertices u, v are in the same connected component of a graph G . This can trivially be done in linear time. Observe that in linear time one can also assign to each vertex of G a *label*, so that the subsequent query between u and v can be answered in constant time by inspecting the label of u and the label of v only.

Understanding the evolution of connected components of a graph when some vertices or edges are removed (or forbid) is of great importance for many applications, in particular for *Emergency Planning* (cf. [4]). Connectivity with forbidden set of edges has been solved almost optimally by [4] in general graphs. However, identifying the connected components in a graph G with a forbidden-set X becomes a challenging question whenever X contains more than two vertices, even if we are allowed to preprocess the graph.

In this paper we consider the problem of labeling the vertices of G such that, for every forbidden subset X of vertices of G , and every vertex u of $G \setminus X$, one can identify the component of u in $G \setminus X$ given the labels of u and the label of each vertex of X .

For this problem and for G planar, Courcelle et al. [3] have proposed a solution using $O(\log n)$ -bit labels and $O(\log n)$ query time, after preprocessing X in time $O(|X|^2)$. We propose an extension of this result to graphs embedded on a surface of Euler genus g (the surface can be orientable or not).

Our contribution

Theorem 1 *Every 3-connected n -vertex Euler genus- g map G admits a $O(g \log n)$ -bit labeling for forbidden-set connectivity queries. All the labels can be computed in time $O(n + g)$. Given a forbidden-set query $X \subset V(G)$, one can preprocess X in time $O(\text{SORT}(|X| + g, n + g))$ so that one can subsequently identify in time $O(\text{PRED}(|X| + g, n + g))$ the connected component of u in $G \setminus X$, for every vertex u of $G \setminus X$.*

Here $\text{SORT}(k, n)$ stands for the time complexity of sorting k integers ranging in $\{1, \dots, n\}$, and $\text{PRED}(k, n)$ for the time complexity for predecessor search in a sequence of k sorted integers in $\{1, \dots, n\}$. It is known that $\text{SORT}(k, n) = O(k\sqrt{\log \log k})$ and $\text{PRED}(k, n) = O(\log \log (n/k))$.

Our scheme is only based on combinatorial embedding (map) of the graph. For the planar case ($g = 0$), our scheme is simpler than the one of Courcelle et al. [3], since it does not require any computational geometry reduction and so avoids the PLANAR POINT LOCATION bottleneck [2]. This results in an exponential improvement of the query time. These time bounds are comparable to those of the recent work of [1]. However, to the best

of our knowledge, this latter scheme, restricted to planar graphs, does not convert into a labeling scheme.

Outline of our construction

First, we compute in $O(n + g)$ time a *cut-graph* C of the surface embedding of the input graph G . Recall that a cut-graph is a subgraph of G such that the surface obtained by cutting along the edges of C is homeomorphic to a disc. We can ensure that the cut-graph C is composed of a spanning tree of bounded degree plus $O(g)$ edges.

By definition of C , cutting along its edges results in a map M whose all the vertices lie on the border of C , i.e., an outerplanar map. Each vertex of G is associated with at most $O(g)$ vertices of M .

We construct a specific forbidden-set connectivity labeling for M and C . For the outerplanar map M we obtain labels of $O(\log(gn)) = O(\log n)$ bits with a scheme much simpler and efficient than the general planar scheme of [3]. For the cut-graph C we develop an ad-hoc scheme with $O(g \log n)$ -bit labels.

Our construction then uses a meta-scheme that allows us to identify the connected component of any vertex u of $G \setminus X$ by querying the components of u in $M \setminus X$ and in $C \setminus X$. The preprocessing of X by our meta-scheme costs $O(|X| + g)$ once X has been sorted.

Acually, our meta-scheme applies to a wider class of graphs, typically the graphs having a *k-page embedding* (cf. [5] for definition) whose spine is constrained to be covered by a forest of at most k trees of degree bounded by k . Labels are of size $O(k \log n)$ bits.

Conclusion

Our solution provides a query time of $O(\text{PRED}(|X|, n))$ (for fixed g), which can be proved optimal even on a path with n vertices by a straightforward reduction to predecessor query. Concerning the label length, our $O(g \log n)$ -bit solution can conceivably be improved. In fact, one can show that $\Omega(\sqrt{g} + \log n)$ -bit labels are required. In the light of our meta-scheme, we leave open the problem of determining the optimal label length for genus- g graphs.

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Constant 2-labelling of weighted cycles

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EXTENDED ABSTRACT

We introduce the concept of constant 2-labelling of a weighted graph. Roughly speaking, a *constant 2-labelling* of a weighted graph is a 2-coloring $\{\bullet, \circ\}$ of its vertex set which preserves the sum of the weight of black vertices under some automorphisms.

The motivation about introducing such labellings comes from covering problems in graphs. These lattest are coverings with balls of constant radius satisfying special multiplicity condition. Let $G = (V, E)$ be a graph and r, a, b be positive integers. A set $S \subseteq V$ of vertices is an (r, a, b) -code if every element of S belongs to exactly a balls of radius r centered at elements of S and every element of $V \setminus S$ belongs to exactly b balls of radius r centered at elements of S . Such codes are also known as (r, a, b) -covering codes [1], (r, a, b) -isotropic colorings [1] or as perfect colorings [5]. For (r, a, b) -codes of the infinite grid with $|a - b| > 4$ and $r \geq 2$, constant 2-labellings help us to give all possible values of a and b .

The notion of (r, a, b) -codes generalizes the notion of domination and perfect codes in graphs. An r -perfect code in a graph is nothing less than an $(r, 1, 1)$ -code. Perfect codes were introduced in terms of graphs by Biggs in [2]. It was shown by Kratochvil [4] that the problem of finding an r -perfect code in graphs (*i.e.*, an $(r, 1, 1)$ -code) is NP-complete. Moreover, this problem is even NP-complete in the case of bipartite graphs with maximum degree three. For more information about perfect codes, see [3, Chapter 11].

Constant 2-labellings. Given a graph $G = (V, E)$, a vertex v of G , a map $w : V \rightarrow \mathbb{R}$ and a subset A of the set $\text{Aut}(G)$ of all automorphisms of G , a *constant 2-labelling* of G is a mapping $\varphi : V \rightarrow \{\bullet, \circ\}$ such that

$$\sum_{\{u \in V \mid \varphi \circ \xi(u) = \bullet\}} w(u) \text{ is constant for all } \xi \in A_\bullet \text{ (respectively } \xi \in A_\circ)$$

where $A_\bullet = \{\xi \in A \mid \varphi \circ \xi(v) = \bullet\}$ (resp. $A_\circ = \{\xi \in A \mid \varphi \circ \xi(v) = \circ\}$). Observe that any coloring using only one color is a constant 2-labelling. Such constant 2-labellings are called *trivial*. Moreover, if φ is a constant 2-labelling of a graph G , then the coloring obtained by exchanging colors is also a constant 2-labelling of G .

We look at weighted cycles with p vertices denoted by \mathcal{C}_p . These vertices $0, \dots, p-1$ have respectively weights $w(0), \dots, w(p-1)$. We will represent such a cycle by the word $w(0) \dots w(p-1)$. Let \mathcal{R}_k denote a k -rotation of \mathcal{C}_p , *i.e.*,

$$\mathcal{R}_k : \{0, \dots, p-1\} \rightarrow \{0, \dots, p-1\} : i \mapsto i + k \bmod p.$$

We set $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and $v = 0$. A coloring $\varphi : \{0, \dots, p-1\} \rightarrow \{\bullet, \circ\}$ of a cycle \mathcal{C}_p is a constant 2-labelling if, for every k -rotation of the coloring, the weighted sum of black vertices is a constant a (resp. b) whenever the vertex 0 is black (resp. white).

We consider eight particular weighted cycles \mathcal{C}_p with at most 4 different weights, namely z, x, y and t . The following words represent respectively cycles of Type 1–8 (see Figure 12) :

$$\begin{aligned} &zx^{p-1}, zx^{\frac{p-2}{2}}tx^{\frac{p-2}{2}}, z(xy)^{\frac{p-1}{2}}, z(xy)^{\frac{p-2}{2}}x, z(xy)^{\frac{p-1}{4}}(yx)^{\frac{p-1}{4}}, \\ &z(xy)^{\frac{p-3}{4}}xx(yx)^{\frac{p-3}{4}}, z(xy)^{\frac{p-2}{4}}t(yx)^{\frac{p-2}{4}}, z(xy)^{\frac{p-4}{4}}xtx(yx)^{\frac{p-4}{4}} \end{aligned}$$

with $x \neq y$ and $p \geq 2$. Note that the exponents appearing in the representation of cycles must be integers. This implies extra conditions on p depending on the type of \mathcal{C}_p . We can give a characterization of all constant 2-labellings of these weighted cycles. For example, if we set $a = \sum_{\{u \in V \mid \varphi \circ \xi(u) = \bullet\}} w(u)$ and $b = \sum_{\{u \in V \mid \varphi \circ \xi'(u) = \bullet\}} w(u)$ for $\xi \in A_\bullet, \xi' \in A_\circ$, then we have the following result for Type 7 cycles.

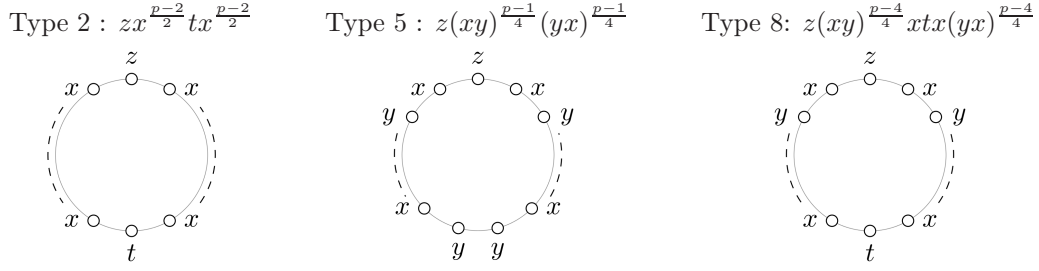


Figure 12: Types of weighted cycles \mathcal{C}_p .

Lemma 1 For cycles \mathcal{C}_p of Type 7, i.e., $z(xy)^{\frac{p-2}{4}}t(yx)^{\frac{p-2}{4}}$ with $t \neq x \neq y$ and $2 < p \in \mathbb{N}$, if φ is a non trivial constant 2-labelling, then φ is either alternate with $a = (\frac{p}{2} - 1)y + z$ and $b = (\frac{p}{2} - 1)x + t$ or $\frac{p}{2}$ -periodic with $a = \alpha(x + y) + t + z$ and $b = (\alpha + 1)(x + y)$ for $\alpha \in \{0, \dots, \frac{p}{2} - 1\}$.

Application to (r, a, b) -codes. We will focus on the graph of the infinite grid \mathbb{Z}^2 . we consider balls defined relative to the Manhattan metric. We can view an (r, a, b) -code of \mathbb{Z}^2 as a particular coloring φ with two colors black and white where the black vertices are the elements of the code. In other words, the coloring φ is such that a ball of radius r centered on a black (respectively white) vertex contains exactly a (resp. b) black vertices.

For 2-colorings of the infinite grid satisfying specific periodicity properties, we will present a *projection and folding method* that associates a weighted cycle to a ball of radius r in \mathbb{Z}^2 . For $r \geq 2$, Puzynina [5] showed that every (r, a, b) -codes of \mathbb{Z}^2 are periodic. Moreover Axenovich [1] gave a characterization of all (r, a, b) -codes of \mathbb{Z}^2 with $r \geq 2$ and $|a - b| > 4$. Using these results, we can apply our method to any (r, a, b) -codes with $r \geq 2$ and $|a - b| > 4$. The weighted cycles obtained by this procedure are of Type 1–8.

So, for $r \geq 2$ and $|a - b| > 4$, there exists an (r, a, b) -code of \mathbb{Z}^2 if and only if there exists a constant 2-labelling of some cycle \mathcal{C}_p , with $v = 0$, $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and a mapping w defined as before, such that

$$a = \sum_{\{u \in V \mid \varphi \circ \xi(u) = \bullet\}} w(u) \text{ and } b = \sum_{\{u \in V \mid \varphi \circ \xi'(u) = \bullet\}} w(u) \quad \forall \xi \in A_\bullet, \xi' \in A_\circ.$$

Hence, the concept of constant 2-labellings allows us to obtain a new characterization of (r, a, b) -codes of \mathbb{Z}^2 with $r \geq 2$ and $|a - b| > 4$. Moreover, we can give all the possible values of constants a and b .

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Colorings and acyclic sets in planar graphs and digraphs

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EXTENDED ABSTRACT

In this extended abstract, we consider the size of acyclic sets in planar graphs and digraphs. Our first motivation is the following well-known conjecture of Albertson and Berman.

Conjecture 1 (Albertson, Berman '79 [1]) *Every n -vertex planar graph contains a set of at least $n/2$ vertices that induces a forest.*

It follows from a result of Borodin [2] that every n -vertex planar graph contains a set of at least $2n/5$ vertices that induces a forest. Salavatipour [12] proved Conjecture 1 for triangle-free planar graphs in the following stronger form.

Theorem 2 (Salavatipour '06 [12]) *Every n -vertex triangle-free planar graph has a set of at least $17n/32$ vertices that induces a forest.*

Stronger results were obtained by Kowalik et al. [7].

One way to generalize the problem of finding large forests in a graph is to think of coloring it with forests. A partition of vertices of a graph G into classes $V_1 \cup \dots \cup V_k$ is an *arboreal partition* if each V_i ($1 \leq i \leq k$) induces a forest in G . A function $f: V(G) \rightarrow \{1, \dots, k\}$ is called an *arboreal k -coloring* if $V_i = f^{-1}(i)$, $i = 1, \dots, k$, form an arboreal partition. The *vertex-arboricity* $a(G)$ of the graph G is the minimum k such that G admits an arboreal k -coloring. The following proposition is an easy consequence of 5-degeneracy of planar graphs.

Proposition 3 *For every planar graph G , $a(G) \leq 3$.*

Clearly, showing that $a(G) \leq 2$ for every planar graph G would imply Conjecture 1 (as well as the Four Colour Theorem). However, there are known to exist planar graphs G with $a(G) = 3$ (see, [3] and [11]).

It is, however, possible to strengthen Proposition 3 as follows.

Theorem 4 (Harutyunyan, Mohar '12 [5]) *Every planar graph has exponentially many 3-arboreal colorings.*

In the same vein, one can also consider large acyclic sets in oriented planar graphs.

Conjecture 5 (Harutyunyan '11 [4]) *Every n -vertex digon-free planar digraph has a set of at least $3n/5$ vertices that induces an acyclic subdigraph.*

Conjecture 5, if true, is best possible, as shown in [4].

As for graphs, one can also color digraphs with acyclic sets. We say that a digraph D is *k -colorable* if $V(D)$ can be colored with at most k colors such that each color class induces an acyclic subdigraph. The minimum k for which such a coloring exists is called the *dichromatic number* of D , and is denoted by $\chi(D)$. This definition was introduced by Neumann-Lara [9], and later, independently, by Mohar [8] when studying circular colorings.

Conjecture 6 (Neumann-Lara '85 [10]) *For every digon-free planar digraph D , $\chi(D) \leq 2$.*

Conjecture 6 seems very difficult and is widely open. The only non-trivial result known to us is the following theorem.

Theorem 7 (Harutyunyan, Mohar '12 [6]) *Every planar digraph D of digirth five has $\chi(D) \leq 2$.*

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On the complexity of the 3-kernel problem in some classes of digraphs

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EXTENDED ABSTRACT

We will denote by D a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. We will denote by $d^+(v)$ the *out-degree* of a vertex v . A *sink* will be a vertex $v \in V(D)$ such that $d^+(v) = 0$. An arc $(u, v) \in A(D)$ is a *symmetric arc* of D if $(v, u) \in A(D)$. The *circumference* of a digraph D is the length of a longest cycle in D , or infinity if there are no cycles in D .

A *biorientation* of a graph G is a digraph obtained from G by replacing each edge $xy \in E(G)$ by either the arc (x, y) or the arc (y, x) or the pair of symmetric arcs $(x, y), (y, x)$. An *orientation* of G is a biorientation without symmetric arcs. A *semicomplete multipartite digraph* is a biorientation of a complete m -partite graph for some $m \geq 2$; a *multipartite tournament* is an orientation of a complete m -partite graph for some $m \geq 2$.

A digraph D is *cyclically m -partite* if there is a homomorphism of D to the directed cycle on m vertices, or equivalently, if there exists a partition (V_0, \dots, V_{m-1}) of the vertices of D such that for every arc $(u, v) \in A(D)$, we have $u \in V_i$ if and only if $v \in V_{i+1} \pmod{m}$. It is easy to see that the length of any cycle in a cyclically m -partite digraph is divisible by m .

A digraph is *t -quasi-transitive* if $(u_0, u_t) \in A(D)$ or $(u_t, u_0) \in A(D)$ whenever (u_0, \dots, u_t) is a path in D . Thus 2-quasi-transitive digraphs are just quasi-transitive digraphs.

A subset N of $V(D)$ is *k -independent* if for every pair of vertices $u, v \in N$, we have $d(u, v), d(v, u) \geq k$; it is *l -absorbent* if for every $u \in V(D) - N$ there exists $v \in N$ such that $d(u, v) \leq l$. A *k -kernel* of D is a k -independent and $(k - 1)$ -absorbent subset of $V(D)$. A 2-kernel is called a *kernel*.

In [1] Chvátal proved that recognizing digraphs that have a kernel is an NP-complete problem; later, Fraenkel proved in [2] that this so-called *kernel problem* remains NP-complete even when restricted to planar digraphs with $d^+ \leq 2, d^- \leq 2$, and $d \leq 3$. We propose to investigate the complexity of the analogous *k -kernel problem*, i.e., the problem of recognizing digraphs that have a k -kernel. Although there are many known sufficient conditions for the existence of k -kernels in digraphs [3, 4], very little is known about the complexity of the k -kernel problem for $k \geq 3$.

A classical result in kernel theory states that every bipartite digraph has a kernel [5]. Since bipartite digraphs are cyclically 2-partite digraphs, it is natural to ask if every cyclically k -partite digraph has a k -kernel. The answer is no. In [6] a simple example of a cyclically 3-partite digraph without a 3-kernel is given. So, the next natural question is to ask for the complexity of the k -kernel problem restricted to the class of cyclically k -partite digraphs. As we have already observed, for $k = 2$ a cyclically 2-partite digraph is simply a bipartite digraph, which always has a kernel [5]. Hence, the 2-kernel problem can be decided in constant time for the family of cyclically 2-partite digraphs. We focus on the complexity of the 3-kernel problem, and its restriction to three natural families of digraphs.

Our main result on cyclically 3-partite digraphs is the following theorem.

Theorem 1 *The 3-kernel problem for the class of cyclically 3-partite digraphs is NP-complete, even when restricted to cyclically 3-partite digraphs of circumference 6.*

As a byproduct of the proof of Theorem 1, we derive the following fact about the kernel problem.

Corollary 2 *The kernel problem restricted to the class of 3-colorable digraphs is NP-complete.*

Also, we observe that, with very simple additional conditions, the 3-kernel problem restricted to the family of cyclically 3-partite digraphs can be decided in constant time. We will denote by \mathcal{Z} the family of cyclically 3-partite digraphs that admit a cyclic 3-partition $D = (V_0, V_1, V_2)$ in which at least one V_i , for $i \in \{0, 1, 2\}$, has no sink.

Theorem 3 *Let D be a digraph, each of the following conditions is sufficient for D to have a 3-kernel.*

1. *Every directed cycle of D has length exactly 3.*
2. *D is a cyclically 3-partite digraph and belongs to \mathcal{Z} .*

We can observe that the first condition of Theorem 3 is not restricted to cyclically 3-partite digraphs. Nonetheless, in the case of cyclically 3-partite digraphs, it follows that the circumference constraint in Theorem 1 cannot be improved.

Depending on the available time, we will also discuss the following results.

Theorem 4 *Let D be a semicomplete multipartite digraph. Then D has a 3-kernel if and only if there is a vertex $v \in V(D)$ such that $\{v\}$ is a 2-absorbing set of $D - (X \setminus \{v\})$, where X is the part of D containing v . As a consequence to this fact, it is obtained that the 3-kernel problem can be decided in polynomial time when restricted to the class of semicomplete multipartite digraphs. Also, if a semicomplete multipartite digraph has a 3-kernel, it can be found in polynomial time.*

A restriction to semicomplete bipartite digraphs leads to improved results.

Theorem 5 *Let $D = (X, Y)$ be a semicomplete bipartite digraph. A 3-kernel of D consists either of all the sinks of D , or of a vertex $v \in X$ such that $Y \rightarrow v$, or of a vertex $v \in Y$ such that $X \rightarrow v$. As a consequence to this fact, it can be determined in linear time whether a semicomplete bipartite digraph has a 3-kernel. Also, if a 3-kernel exists, it can be found in linear time.*

Finally, we prove what seems to be the first two cases of a more general result in k -quasi-transitive digraphs.

Theorem 6 *Let $2 \leq k \leq 3$ be an integer. The k -kernel problem restricted to the class of k -quasi-transitive digraphs can be decided in polynomial time. Also, if a k -kernel exists, it can be constructed in polynomial time.*

Further problems and directions will be also discussed.

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On k -critical P_5 -free graphs

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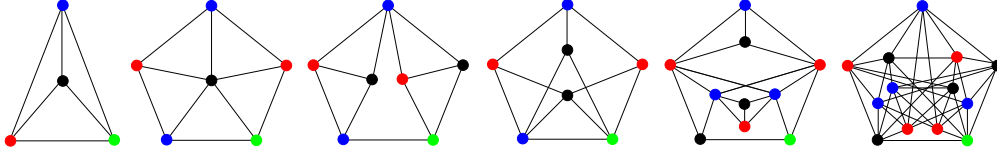
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EXTENDED ABSTRACT

The CHROMATIC NUMBER problem for P_5 -free graphs is known to be NP-hard [6]. However for fixed k , the k -colorability question for P_5 -free graphs can be answered in polynomial time [4, 5]. More generally, the k -colorability question for P_t free graphs has been well studied [10, 9, 7, 1, 4, 5, 8], with the cases when $t < 5$ being relatively trivial. The following table highlights the current state of the research for connected graphs.

$k \backslash t$	3	4	5	6	7	8	9	10	11	...
3	$O(m)$	$O(m)$	$O(m)$	$O(mn^\alpha)$?	?	?	?	?	...
4	$O(m)$	$O(m)$	P	?	?	?	NP_c	NP_c	NP_c	...
5	$O(m)$	$O(m)$	P	?	?	NP_c	NP_c	NP_c	NP_c	...
6	$O(m)$	$O(m)$	P	?	NP_c	NP_c	NP_c	NP_c	NP_c	...
7	$O(m)$	$O(m)$	P	?	NP_c	NP_c	NP_c	NP_c	NP_c	...
...

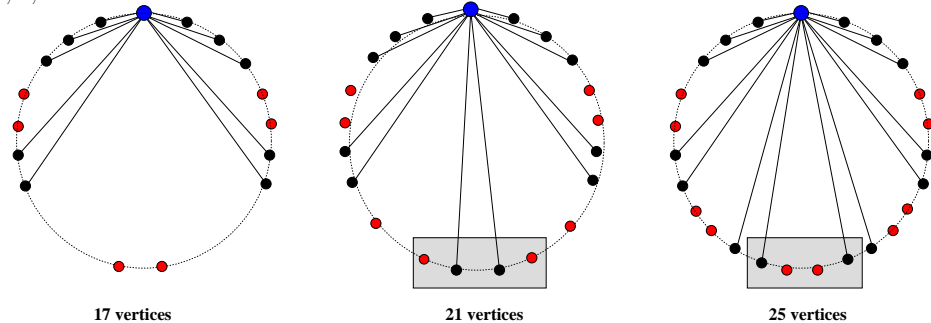
The polynomial time algorithms for answering the k -colorability question for P_5 -free graphs will return a valid k -coloring if one exists, but otherwise does not provide a *no-certificate* – or a minimal obstruction that makes the graph non k -colorable. This motivates the study of k -critical P_5 -free graphs. We define a graph G , with chromatic number k , to be *k -critical with respect to a graph class \mathcal{C}* if every proper subgraph of G belonging to \mathcal{C} is $k - 1$ colorable. It has been shown that when \mathcal{C} is the class of P_5 -free graphs, exactly 6 are 4-critical as illustrated below [2].



These graphs have been subsequently used to answer the 3-colorability question for P_5 -free graphs in linear time [8]. We conjecture that the following construction produces an infinite set of 5-critical graphs with respect to the class of P_5 -free graphs.

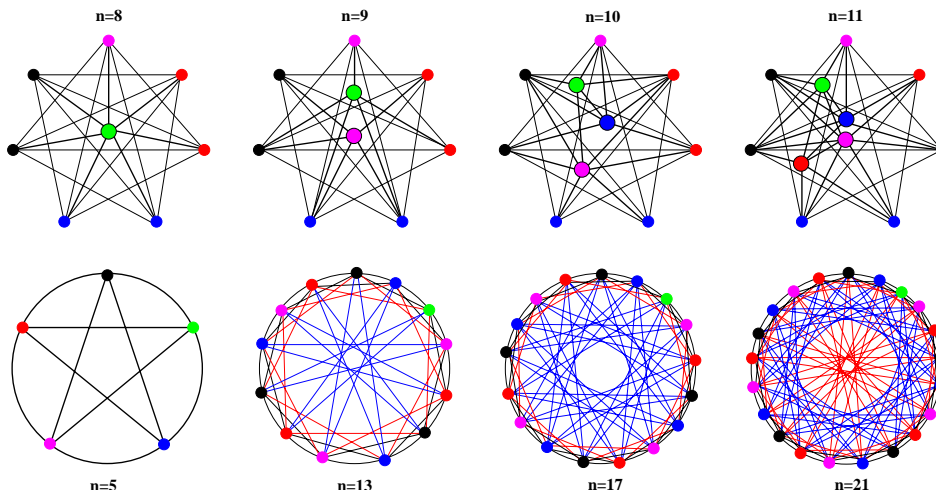
Regular construction: Consider $4(j+2)+1$ vertices on a circle for $j \geq 1$. Each vertex is adjacent to its 3 closest neighbours on each side. Then non-adjacent to the next 2, then adjacent to the next 2, then non-adjacent to the next 2, then adjacent to the next 2, and so on.

As an example, the following graphic illustrates the regular neighbourhood structure for $j = 2, 3, 4$.



It is not difficult to show that the graphs produced by this construction are $2K_2$ -free (and hence P_5 -free) and that their chromatic number is 5. However, proving the graphs are 5-critical (w.r.t. the class of P_5 -free graphs) is currently a work in progress.

It is interesting to note that for $j > 4$, the constructed graphs will contain a C_5 by considering vertices at relative positions $[-10, -5, 0, 5, 10]$ on the cycle. When \mathcal{C} is the class of (P_5, C_5) -free graphs it is obvious that K_5 will be 5-critical. Thus, by the Strong Perfect Graph Theorem [3], all other 5-critical graphs must contain a $\overline{C_7}$ as an induced subgraph. Using this as a starting point we found via computer search an additional 7 graphs that were 5-critical, as illustrated below.



Using non-trivial strategies, our algorithm terminates, which demonstrates that these are the only 8 graphs that are 5-critical with respect to the class of (P_5, C_5) -free graphs.

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A note on strong edge-colouring

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EXTENDED ABSTRACT

In this note the graphs we consider are finite, without multiple edges and without loops. A proper edge-colouring of a graph $G = (V, E)$ is an assignment of colours to the edges of the graph such that two adjacent edges do not use the same colour. A *strong edge-colouring* (called also distance 2 edge-colouring) of a graph G is a proper edge-colouring of G , such that every edge of a path of length 3 $uvxy$ uses a different colour ($c(uv) \neq c(vx)$, $c(vx) \neq c(xy)$ and $c(uv) \neq c(xy)$). We denote by $\chi'_s(G)$ the *strong chromatic index* of G which is the smallest integer k such that G can be strong edge-coloured with k colours.

The notion of strong edge-colouring was introduced by Fouquet and Jolivet in 1983 [4, 5]. Strong edge-colouring can be used to model the conflict-free channel assignment in radio networks [2, 7, 8, 9, 10, 11].

Let Δ denote the maximum degree of a graph. In 1985, during a seminar in Prague, Erdős and Nešetřil gave a construction of graphs having strong chromatic index equal to $\frac{5}{4}\Delta^2$ when Δ is even and $\frac{1}{4}(5\Delta^2 - 2\Delta + 1)$ when Δ is odd. They conjectured that the strong chromatic index is bounded by these values and it was verified for $\Delta \leq 3$ [1, 6]. In [3] Faudree *et al.* proved that $\chi'_s(G) \leq 4\Delta + 4$ for planar graphs with maximum degree $\Delta \geq 3$. Moreover they conjectured:

Conjecture 1 (Faudree *et al.* [3]) *If G is a planar subcubic graph, then $\chi'_s(G) \leq 9$.*

If the conjecture is true, then the upper bound is best possible (since the prism P has $\chi'_s(P) = 9$).

Let $\text{mad}(G)$ be the maximum average degree of the graph G *i.e.*

$$\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$$

where $V(H)$ and $E(H)$ denote the sets of vertices and edges of H , respectively.

In this talk we study the bounds of the strong chromatic index of subcubic graphs considering their maximum average degree and we prove the following results:

Theorem 2 *Let G be a subcubic graph:*

1. *If $\text{mad}(G) < \frac{15}{7}$, then $\chi'_s(G) \leq 6$.*
2. *If $\text{mad}(G) < \frac{27}{11}$, then $\chi'_s(G) \leq 7$.*
3. *If $\text{mad}(G) < \frac{13}{5}$, then $\chi'_s(G) \leq 8$.*
4. *If $\text{mad}(G) < \frac{36}{13}$, then $\chi'_s(G) \leq 9$.*

For cases 1, 2 and 4, the given upper bounds on the maximum average degree are sharp: there exist graphs with $\text{mad}(G) = \frac{7}{3}$ (resp. $\frac{5}{2}, \frac{20}{7}$) which are not strong edge-colourable with 6 (resp. 7, 9) colours.

For planar graphs it follows (we recall that the *girth* of a graph is the size of a shortest cycle in this graph):

Corollary 3 *Let G be a planar subcubic graph with girth g :*

1. *If $g \geq 14$, then $\chi'_s(G) \leq 6$.*
2. *If $g \geq 10$, then $\chi'_s(G) \leq 7$.*

3. If $g \geq 8$, then $\chi'_s(G) \leq 8$.

4. If $g \geq 7$, then $\chi'_s(G) \leq 9$.

In this talk we are also interested in finding a bound for the strong chromatic index of planar subcubic graphs. We improve the fourth part of Corollary 3 and give a partial answer to Conjecture 1, by showing the following:

Theorem 4 *Let G be a planar subcubic graph containing neither induced 4-cycles, nor induced 5-cycles. Then $\chi'_s(G) \leq 9$.*

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Structures in Cubic Graphs

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EXTENDED ABSTRACT

We present essentially two results where the second one is joint work with J.Hägglund. Both results disprove several conjectures which are related to the circuit double cover conjecture; see [2, 3, 4, 5].

Firstly, we show that a set S of 2-connected graphs must satisfy several conditions if the following holds for S :

Every 3-connected cubic graph G has a matching M such that every component L of $G - M$ is a subdivision of an element of S and L has an even number of vertices of degree 2.

(For our purpose, it suffices to assume that S has only cubic graphs and even circuits as elements.) For instance, we prove that S must contain a cubic graph which is not hamiltonian. This approach generalizes the concept of component factors in cubic graphs.

Secondly, we present the only known construction of an infinite family of cyclically 5-edge connected permutation snarks; i.e. snarks which have a 2-factor consisting of two chordless circuits. Note that the Petersen graph has been claimed for a long time to be the only cyclically 5-edge connected permutation snark which has recently been disproved in [1].

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Minimum number of palettes in edge colorings

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EXTENDED ABSTRACT

Let G be a finite simple graph, let C be a set of *colors* and let $f : E(G) \rightarrow C$ be an edge-coloring of G . We shall always assume that f is *proper*, i.e., any two adjacent edges get distinct colors. The *palette* of a vertex $v \in V(G)$ with respect to f is the set $S_f(v)$ of colors of edges incident to v .

Two vertices of G are distinguished by a coloring f if their palettes are distinct, i.e., $S_f(u) \neq S_f(v)$. There are many papers dealing with distinguishing either all or only neighboring vertices in a graph. In this talk we are interested in minimizing the number of distinguished vertices, i.e., in the minimum number of palettes taken over all possible proper (edge-) colorings of a graph. For a given graph G , we denote this number by $\check{s}(G)$ and call the *palette index* of G .

The minimum number of colors required in a proper coloring of a graph G is called the *chromatic index* of G and is denoted by $\chi'(G)$. Recall that, by Vizing's theorem, the chromatic index of G equals either $\Delta(G)$ or $\Delta(G) + 1$. A graph with $\chi'(G) = \Delta(G)$ is called class 1, while a graph with $\chi'(G) = \Delta(G) + 1$ is called class 2. Our first result on the palette index is (almost) obvious.

Proposition 1 *The palette index of a graph G is 1 if and only if G is regular and class 1.*

A proper coloring of G using $\chi'(G)$ colors is called *minimum*. In general, however, minimum colorings do not provide the minimum number of palettes. For instance, let us consider the case of complete graphs.

Since the graphs K_1 and K_{2k} are class 1, we have $\check{s}(K_1) = \check{s}(K_{2k}) = 1$. On the other hand, it is easy to see that the minimum coloring of K_n for odd n induces n distinct palettes. Indeed, each palette has $n - 1$ colors. That means that at each vertex exactly one color is missing. Further, since n is odd, each color misses at least one vertex. Consequently, each color misses exactly one vertex and missing colors are distinct for distinct vertices.

However, by increasing the number of colors we can reach the number of palettes 3 or 4. The aim of the talk is to determine palette indices of complete graphs:

Theorem 2

$$\check{s}(K_n) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod{2} \text{ or } n = 1 \\ 3, & \text{if } n \equiv 3 \pmod{4} \\ 4, & \text{if } n \equiv 1 \pmod{4} \end{cases}$$

Relation between Graphs

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EXTENDED ABSTRACT

Graphs are frequently employed to model natural or artificial systems. A good example is protein-protein interaction networks (PPIs) in biology. Proteins contain domains that determine their function. Information on the domain content of proteins, which can be seen as a relation $R \subseteq D \times P$ between domains and proteins which contain them is readily available from biological databases. Candidates for interaction between domains can be derived from knowledge on the interacting proteins, i.e., a graph G with vertex set P , and the relation R : If $(p, p') \in P$, $(d, p) \in R$, and $(d', p') \in R$, then the domains (d, d') are putative interaction partner.

More abstractly, let $G = (V_G, E_G)$ be a graph, B a finite set, and $R \subseteq V \times B$ a binary relation, so that, for every element $b \in B$, we can find an element $v \in V_G$ such that $(v, b) \in R$. Then the graph $G * R$ has vertex set B and edge set

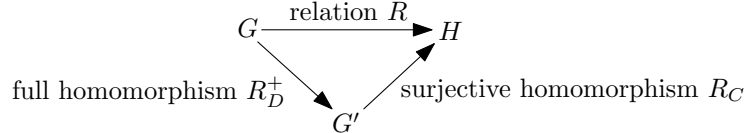
$$E_{G*R} = \{(u, v) \in B \times B \mid \text{there is } (x, y) \in E_G \text{ and } (x, u), (y, v) \in R\}.$$

If there exists an R satisfying the equation $G * R = H$, we say there is a *relation* from G to H . Note that graphs G and H can be seen as conjugate, i.e., $G = R^+ \circ H \circ R$.

This concept has a close relation with a well studied graph theory concept: graph homomorphism. For given graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ a *homomorphism* $f : G \rightarrow H$ is a mapping $f : V_G \rightarrow V_H$ such that $(u, v) \in E_G$ implies $(f(u), f(v)) \in E_H$. It naturally defines a mapping $f^1 : E_G \rightarrow E_H$ by setting $f^1((x, y)) = (f(x), f(y))$ for all $(x, y) \in E_G$. If both f and f^1 are surjective, we call f a *surjective homomorphism*. Note that all surjective homomorphisms $f : G \rightarrow H$ are (modulo representation) also relations $G * R = H$ where $R = \{(u, f(u), x \in V_G)\}$.

Obviously, relations that are not functional are not homomorphisms, but corresponds to the multihomomorphisms. A relation with full domain thus can be regarded as a *surjective multihomomorphism*, a multihomomorphism such that pre-image of every vertex in H is non-empty and for every edge (u, v) in H we can find an edge (x, y) in G satisfying $u \in f(x)$, $v \in f(y)$.

Graph relations compose, i.e., $(G * R) * S = G * (R \circ S)$. We also show that every relation R can be decomposed, in a standard way, to a relation R_D duplicating vertices and a relation R_C contracting vertices. In diagram form, this is expressed as:



This further develops the connection in between relations and homomorphisms. For example, the computational complexity of deciding the existence of a relation from G to a given H lies in between the complexity of the corresponding problems of existence of homomorphism and surjective homomorphism.

Let \leq_P^{Tur} indicate polynomial time Turing reduction.

Theorem 1 For finite H our relation problem sits in the following relationship:

$$\text{HOM}(H) \leq_P^{\text{Tur}} \text{FUL-REL}(H) \leq_P^{\text{Tur}} \text{SUR-HOM}(H). \quad (2)$$

Here $\text{FUL-REL}(H)$ is *full relation problem* i.e., for a fixed graph H , take as input some finite G and ask whether there is a relation with full domain from G . $\text{HOM}(H)$ denoted as *homomorphism problem* which takes as input some finite G and asks whether there is a homomorphism from G to H . The *surjective homomorphism problem* $\text{SUR-HOM}(H)$ takes as input some finite G and asks whether there is a surjective homomorphism from G to H .

There are three different ways how to define a notion similar to graph cores for relations between graphs.

Cores of graph are minimal representatives in the corresponding equivalence class of homomorphism equivalence. We consider cores of relational equivalence classes. Two graphs G and H are *relationally equivalent*, $G \sim H$, if there are relations R and S such that $G * R = H$ and $H * S = G$. If we require relations R and S to be full, we denote the equivalency as $G \sim_f H$. A graph is an *R-core* if it is the smallest graph (in the number of vertices) in its equivalence class under \sim_f . We show that if G is an R-core, then every relation R such that $G * R = G$ satisfies the Hall condition and thus contains a monomorphism. And R-core is unique up to isomorphism. An R-core of a graph G , $G_{\text{R-core}}$, is the smallest graph in the same equivalence class of G . We prove that $G_{\text{R-core}}$ is isomorphic to an induced subgraph of G .

Proposition 2 $G_{\text{R-core}}$ can be characterized by an algorithm, which removes all vertices $v \in G$ such that

- (1) the neighborhood of v is union of neighborhood of some other vertices and
- (2) there is vertex u , $u \neq v$, such that $N_G(v) \subseteq N_G(u)$.

Because every subgraph has a homomorphism to the original graph, cores of graph (in the sense of homomorphism) are defined as the minimal graphs which do not have homomorphism to its subgraph. Similarly, we consider the relation from a graph to its subgraph. Let H be a subgraph of G , an *R-retraction* of G to H is a relation R such that $G * R = H$ and $(x, x) \in R$ for all $x \in V_H$. A graph is *R-reduced* if there is no R-retraction to a proper subgraph. We show that the R-reduced graph of a graph is unique up to isomorphism. Thus, we can also speak about “the R-reduced graph of a graph G ” as the smallest subgraph on which it can be retracted. We also show that

Proposition 3 a graph is R-reduced if and only if it is a graph core.

Finally, we introduce another notion related to a graph core that is meaningful only for graph relations. An *R-coretraction* of subgraph H to G is a relation R such that $H * R = G$ and $(x, x) \in R$ for all $x \in V_H$. A graph G is an *R-cocore* if there is no proper subgraph of G that is an R-coretract of G . We show the following proposition and consequently R-cocores are unique up to isomorphism and can be computed in polynomial time. We also show that the R-cocore is the core of equivalence class under \sim .

Proposition 4 G is an R-cocore if and only if any vertex neighborhood is not the union of other vertex neighborhoods.

For the simplest case $G = H$, a characterization of solutions of the equation $G * R = G$ is given. A relation R is an *automorphism* of G if it is of the form $R = \{(x, f(x)) | x \in V_G\}$ and $f : V_G \rightarrow V_G$ is an automorphism of G . We show that:

Theorem 5 All solutions of $G * R = G$ are automorphisms if and only if G has property that the neighborhoods of vertices do not contain each other.

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Symmetry breaking in graphs

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EXTENDED ABSTRACT

This talk is concerned with automorphism and endomorphism breaking of finite and infinite graphs.

Albertson and Collins [1] introduced the *distinguishing number* $D(G)$ of a graph G as the least cardinal d such that G has a labeling with d labels that is only preserved by the trivial automorphism.

This concept has spawned a wealth of papers, mostly on finite, but also on infinite graphs. We are interested in the infinite motion conjecture of Tom Tucker. In particular, we present partial results pertaining to the conjecture, a generalization to uncountable graphs, and results that support the conjecture.

Moreover, we extend this concept to endomorphisms and introduce the *endomorphism distinguishing number* $D_e(G)$ of a graph G as the least cardinal d such that G has a vertex labeling with d labels that is only preserved by the trivial endomorphism.

As the number of endomorphisms usually vastly exceeds the number of automorphisms, the new concept opens challenging problems, several of which will be presented. Moreover, we extend results about the distinguishing number of finite and infinite graphs to the endomorphisms distinguishing number. The starting point is the

Lemma 1 (Motion Lemma, Russell and Sundaram [5]) *Let the motion $m(G)$ of a graph G be the minimum number of vertices that is moved by any nonidentity automorphism of G . Then*

$$2^{\frac{m(G)}{2}} \geq |\text{Aut}(G)|$$

implies $D(G) \leq 2$.

The immediate generalization to infinite graphs is the Motion Conjecture, with Tucker's Infinite Motion Conjecture as a special case:

Conjecture 2 (Motion Conjecture [2]) *Let G be a connected, infinite graph with infinite motion $m(G)$. Then $2^{m(G)} \geq |\text{Aut}(G)|$ implies $D(G) \leq 2$.*

Conjecture 3 (Tucker's Infinite Motion Conjecture [6]) *Let G be a connected, locally finite infinite graph with infinite motion $m(G)$. Then $D(G) \leq 2$.*

Both conjectures are open, but partial results and connections to other structures will be presented in the talk. We list two of them below. The first one is by Florian Lehner.

Theorem 4 *The Infinite Motion Conjecture is true for graphs of growth*

$$O\left(2^{(1-\epsilon)\frac{\sqrt{n}}{2}}\right)$$

Another one is from Cuno, Imrich and Lehner [2].

Theorem 5 *Let G be an infinite, connected graph with infinite motion. If $m(G) \geq |\text{Aut}(G)|$, then $D(G) \leq 2$.*

For countable G this is easily shown by induction, for graphs G with larger cardinality by transfinite induction. We have the following corollary.

Corollary 6 *Let G be an uncountably infinite, connected graph with infinite motion, and suppose that $2^{m(G)} > |Aut(G)|$. Then, under the assumption of the general continuum hypothesis, $D(G) \leq 2$.*

For the analogous result for countably infinite, connected graphs one assumes the CH. However, if the graphs are locally finite, then the CH is not needed. This was shown by Simon Mark Smith (see [4]) and follows from results of either Halin, Trofimov or Evans.

For the endomorphism distinguishing number we have analogues to most of the above, see Imrich et al. [3]. Let us just state the analogues to the Motion Conjecture, Theorem 5, and Corollary 6.

Conjecture 7 (Endomorphism Motion Conjecture [3]) *Let the endomorphism motion $m_e(G)$ of a graph G be the minimum number of vertices that is moved by any nonidentity endomorphism of G , and let G be a connected, infinite graph with infinite $m_e(G)$. Then $2^{m_e(G)} \geq |End(G)|$ implies $D_e(G) \leq 2$.*

Theorem 8 *Let G be a connected infinite graph with infinite endomorphism motion. If $m_e(G) \geq |End(G)|$, then $D_e(G) \leq 2$.*

Corollary 9 *Let G be a connected, uncountable infinite graph with infinite endomorphism motion, and suppose that $2^{m_e(G)} > |End(G)|$. Then, under the assumption of the general continuum hypothesis, $D_e(G) \leq 2$.*

Just as the Motion Conjecture and Tucker's Infinite Motion Conjecture, Conjecture 7 is still open. However, there are many classes of graphs that support these conjectures. For example, countable infinite trees T without leaves have infinite endomorphism motion $m_e(T)$, uncountably many endomorphisms, and

$$2^{m_e(T)} \geq |End(T)|.$$

In accordance with Conjecture 7 they have endomorphism distinguishing numbers ≤ 2 , see [3].

Notice that the extension of this result to uncountable trees is still open, just as, for example, the endomorphism distinguishing numbers of hypercubes (of dimension > 3) have not yet been determined.

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Cyclic words and vertex colourings of plane graphs

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EXTENDED ABSTRACT

Let $\mathbb{A} = \{a, b, c, \dots\}$ be a finite alphabet, whose element are called letters (digits, colours, symbols, ...).

The *word* of length n over \mathbb{A} is an expression $w = a_1a_2 \dots a_n$, where $a_i \in \mathbb{A}$ for all $i = 1, 2, \dots, n$. *Subword* \bar{w} of the word w is an expression $\bar{w} = a_i a_{i+1} \dots a_j$ with $1 \leq i \leq j \leq n$.

The *cyclic word* of length n is an expression $w = a_1a_2 \dots a_n$, $n \geq 2$ (Consider the cyclic word as a sequence of consecutive labells on the vertices of a cycle of length n). A subword of a cyclic word is its arbitrary part.

Let us recall some properties that words can have. A word is *proper* if it no two consecutive letters are the same. For example the word "abcba" is proper but the word "abbc" is not proper.

The word $a_1a_2 \dots a_n$, $n \geq 1$, is simple if $a_i \neq a_j$ for $i \neq j$.

The word of the form $a_1a_2 \dots a_{2k}$ with property that $a_i = a_{i+k}$ for all $i = 1, 2, \dots, k$ is called the *repetition*. A word is called *nonrepetitive* if none of its subwords is a repetition.

Examples: The word "abcabc" is a repetition, the word "abcacbd" is nonrepetitive, while the word "abcabcdea" is not nonrepetitive because it contains a subword "bcbcb", which is a repetition.

A *palindrom* is any word which can be read in the same way from the front and from the back. The word is *palindromfree* if no its subword is a palindrom.

Examples: The words "abcdcda" and "abcddca" are palindroms while the word "abcdbda" is palindrom free.

A word is a *weak parity* one if at least one letter in it appears there an odd number of times. A word is a *strong parity* one if each used letter in it is used there an odd numbers of times. For example the word "abcabde" is a weak parity word. The word "abcabdadb" is a strong parity word.

Consider a 2-connected plane graphs. All its faces are bounded by cycles, called the *facial cycles*. If we label all vertices of a 2-connected plane graph G with letters from an alphabet \mathbb{A} , then any face $\alpha = [v_1, v_2, \dots, v_k]$ determined by the vertices v_1, v_2, \dots, v_k can be associated with a cyclic word $a_1a_2 \dots a_k$, where k is size (degree) of the face α and a_i is a labell of the vertex v_i . The word $a_1a_2 \dots a_k$ is called the facial word of the face α of G .

In our talk we will consider the following problem:

Problem: What is the minimum number of letters in an alphabet \mathbb{A} that allows to label the vertices of a given 2-connected plane graph G in such a way that all the facial words of G over \mathbb{A} have a given property \mathcal{P} ?

We will give a survey on results and open questions concerning this problem for several properties of words.

Claw-free cubic graphs and their global secure sets

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EXTENDED ABSTRACT

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set if every vertex $v \in V - D$ has a neighbour in D . The cardinality of a minimum dominating set in a graph G is denoted by $\gamma(G)$. Many variants of dominating sets were investigated e.g. connected dominating sets, total dominating sets, k -dominating sets. Recall that a set TD is a total dominating set if every vertex is adjacent to a vertex in TD . Let $\gamma_t(G)$ denote the total domination number, i.e., minimum cardinality of a total dominating set in a graph G .

Another interesting type of dominating sets are global secure sets. We say that a set $SD \subseteq V$ is a global secure set if $N[SD] = V$ and $|N(X) \cap SD| \geq |N[X] - SD|$ for any $X \subseteq SD$ [2]. A global security number, denoted by $\gamma_s(G)$, is the cardinality of a minimum global secure set of a graph G .

From the definition of global secure sets it follows that $\gamma_s(G) \geq \lceil |V|/2 \rceil$ for any graph G . On the other hand it was proven that if a minimum vertex degree in a graph G is greater or equal to 3, then $\gamma_t(G) \leq |V|/2$ [1]. Moreover, Henning et al. [4] characterized graphs which achieve the equality in the above bound. It turns out that all these graphs are cubic. Moreover, they achieve the equality in the lower bound on global security number. For claw-free cubic graphs of order $n \geq 10$ it was shown that $\gamma_t(G) \leq 4n/9$ [8]. If $n < 10$, then there are exactly two claw-free cubic graphs with total domination number one-half their order: K_4 and a graph presented in Fig.1. For these two graphs also the global security number equals one-half their order.

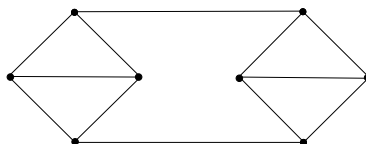


Figure 13: The graph with total domination number one-half its order.

Global secure sets were investigated for grid-like graphs [3], cactus trees [5] and cographs [6, 7]. Now we present new results concerning minimum global secure sets in claw-free cubic graphs.

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Identification of sets of edges using factors

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EXTENDED ABSTRACT

Identifying codes in graphs have been studied for more than a decade (see [2]). In the class of line graphs these codes were studied in [1]. There the problem is equivalent to locating an arbitrary edge with the aid of a suitable subset of edges (like a perfect matching). Often in identification we are also interested in locating more than one object. In this paper, we concentrate on this, i.e., finding optimal structures to locate sets of edges. Let us now introduce the problem in details together with some notation.

We assume that $G = (V, E)$ is a finite simple graph, which is also connected and undirected. The girth of G is denoted by $g = g(G)$ and the minimum degree by $\delta = \delta(G)$. Let $e \in E$ be an edge. We use the notation $N[e]$ for the set of edges adjacent to e (including also e). Similarly, for a set of edges $S \subseteq E$, we write:

$$N[S] = \bigcup_{e \in S} N[e].$$

A nonempty subset of E is called a *code* in G and its elements are called *code-edges*. Let $C \subseteq E$ be a code and e an edge in E . An *identifying set* of the edge e with respect to C is defined via $I[e] = N[e] \cap C$. For a subset $S \subseteq E$ we use analogous notation $I[S] = N[S] \cap C$.

Definition 1 Let ℓ be a positive integer. A code $C \subseteq E$ is called ℓ -edge-identifying if

$$I[S_1] \neq I[S_2]$$

for all distinct subsets S_1 and S_2 of E with cardinalities of at most ℓ .

This gives the concept of *edge-identifying codes* defined in [1] when $\ell = 1$. The smallest cardinality of an ℓ -edge-identifying code in G is denoted by $\gamma_\ell^{\text{EID}}(G)$ and a code attaining this cardinality is called *optimal*.

Classification

Next we classify all those graphs which admit 2-edge-identifying codes and show that there does not exist non-trivial graphs which admit an ℓ -identifying code for $\ell \geq 3$ (actually, we show a stronger result related to claw-free graphs). Let us introduce two properties, which we will use in the sequel:

- (P5) If C_5 is a 5-cycle in G , then there exists at most one vertex in C_5 of degree 2 (that is, all the other vertices are incident with at least one edge of E not belonging to C_5),
- (P6) If C_6 is a 6-cycle in G , then there exist two vertices in C_6 , say u and v , of degree at least three such that $d(u, v) \leq 2$ in C_6 .

Theorem 2 (i) A graph on at least three vertices admits a 2-edge-identifying code if and only if $\delta \geq 2$, $g \geq 5$, any 5-cycle has the property P5 and any 6-cycle has the property P6. (The only graph on two vertices admitting 2-edge-identifying code is the complete graph K_2).

(ii) Let $\ell \geq 3$. Apart from $G = K_2$, there do not exist any connected graphs admitting an ℓ -edge-identifying code.

Lower bounds, constructions and algorithms

We begin by giving a useful bound in terms of the order of a graph.

Theorem 3 For a graph $G = (V, E)$ (other than K_2), we have $\gamma_2^{\text{EID}}(G) \geq |V|$.

We say that a subgraph H of G is a k -factor if H contains all the vertices of G and all the vertices of H are of degree k . The k -factors of graphs have been widely studied and for the known results we refer to the survey [3]. The previous lower bound is sharp for infinitely many graphs as is shown in the following theorem.

Theorem 4 Let $G = (V, E)$ be graph.

- (i) If G has 2-factor and $g(G) \geq 7$, then the edges of any 2-factor form an optimal 2-edge-identifying code in G with cardinality $|V|$.
- (ii) If G is Hamiltonian, $|V| \geq 7$ and $g(G) \geq 6$, then the edges of any Hamilton cycle form an optimal 2-edge-identifying code in G with cardinality $|V|$.
- (iii) If G has a 3-factor and admits a 2-edge-identifying code, then the edges of any 3-factor form a 2-edge-identifying code with cardinality $3|V|/2$.

However, not all graphs have 2-edge-identifying code of size $|V|$. For example, the optimal 2-edge-identifying code of the Petersen graph (with 10 vertices) has cardinality 12.

We provide an algorithm for graphs with girth at least six showing that the size of the optimal 2-edge-identifying code is bounded above by two times the number of vertices.

Theorem 5 If $G = (V, E)$ is a graph with $g(G) \geq 6$ admitting a 2-edge-identifying code, then we have $\gamma_2^{\text{EID}}(G) \leq 2|V|$.

Moreover, we can construct an infinite family of graphs $G_i = (V_i, E_i)$ (with $g(G_i) = 6$, $\forall i = 1, 2, \dots$) such that the cardinality of an optimal 2-edge-identifying code in G_i approaches $2|V_i|$ as i tends to infinity.

The next theorem gives a bound in terms of the size of a graph.

Theorem 6 If C is a 2-edge-identifying code in a graph G consisting of k edges, then we have $|E| \leq k^{\frac{3}{2}}$. On the other hand, there exists an infinite family of graphs $G_\nu = (V_\nu, E_\nu)$ such that they admit a 2-edge-identifying code of cardinality k_ν with $|E_\nu| = \Theta(k_\nu^{\frac{3}{2}})$.

We have the following counter-intuitive result stating that increasing the number of edges, i.e., the number of identified objects, in a graph with girth at least six does *not* increase the number of edges needed in the optimal 2-edge-identifying code.

Theorem 7 Let $G = (V, E)$ be a graph admitting a 2-edge-identifying code such that the girth of G is at least six and $G \neq K_2$. Assume that G' is a graph obtained from G by adding such an edge e that the girth of graph G' remains at least six. Then G' admits a 2-edge-identifying code and we have

$$\gamma_2^{\text{EID}}(G) - 3 \leq \gamma_2^{\text{EID}}(G') \leq \gamma_2^{\text{EID}}(G).$$

Moreover, there exist graphs attaining these bounds.

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Almost all fullerene graphs are almost Hamiltonian

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EXTENDED ABSTRACT

Fullerene graphs, i.e. cubic planar 3-connected graphs with pentagonal and hexagonal faces, are conjectured to be Hamiltonian. This is a special case of a conjecture of Barnette. We show that if the pentagonal faces of a fullerene graph G are far enough from each other, then G contains a Hamiltonian cycle.

d -Strong total colorings of graphs

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EXTENDED ABSTRACT

If $c : V \cup E \rightarrow \{1, 2, \dots, k\}$ is a proper total coloring of a graph $G = (V, E)$ then the *palette* $S[v]$ of a vertex $v \in V$ is the set of colors of the incident edges and the color of v : $S[v] = \{c(e) : e = vw \in E\} \cup \{c(v)\}$. A total coloring c *distinguishes* vertices u and v if $S[u] \neq S[v]$. A d -strong total coloring of G is a proper total coloring that distinguishes all pairs of vertices u and v with distance $1 \leq d(u, v) \leq d$. The minimum number of colors of a d -strong total coloring is called d -strong total chromatic number $\chi_d''(G)$ of G . Such colorings generalize strong total colorings and adjacent strong total colorings as well.

The d -strong total chromatic number is monotonous with respect to the distance and additive but not hereditary (for example, $\chi_2''(C_5) = 5$ but $\chi_2''(C_5 + e) = 4$).

Let n_i denote the maximum number of vertices of degree i that are of pairwise distance at most d and let $\mu_d''(G) = \max\{\min\{j : \binom{j}{i+1} \geq n_i\} : \delta(G) \leq i \leq \Delta(G)\}$. Obviously, $\mu_d''(G)$ is a lower bound for $\chi_d''(G)$.

Zhang et al. [2] conjectured that the d -strong total chromatic number attains one of two possible values if $d \geq 2$.

Conjecture 1 $\chi_d''(G) \leq \mu_d''(G) + 1$ for all graphs G and $d \geq 2$.

We prove that this conjecture is not true in general. Moreover, we show that the difference between the d -strong total chromatic number $\chi_d''(C_n)$ of a cycle C_n and the lower bound $\mu_d''(C_n)$ may be arbitrarily large.

Theorem 2 Let k and s be positive integers such that $k > 7s + 2$. If $d = \binom{k}{3} - 1$ and $n = 3\binom{k}{3} - 1$ then $\mu_d''(C_n) = k$ and $\chi_d''(C_n) \geq k + s$.

Moreover, we determine some general bounds for $\chi_d''(G)$, determine $\chi_d''(P_n)$ completely for paths P_n , give some exact values for the d -strong total chromatic number of cycles (up to $d = 16$), and present results for circulant graphs.

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\$(r, s)\$-Fractional \$(\mathcal{P}, \mathcal{Q})\$-Total Colorings of Graphs

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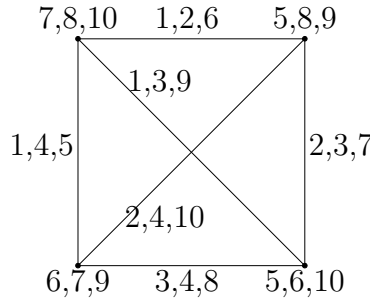
EXTENDED ABSTRACT

We denote the class of all finite simple graphs by \mathcal{I} (see [1]). A *graph property* \mathcal{P} is any non-empty isomorphism-closed subclass of \mathcal{I} . A property \mathcal{P} of graphs is called *hereditary* if it is closed under taking subgraphs, i. e., $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$. A property \mathcal{P} is called *additive* if it is closed under disjoint union of graphs, i. e., $G \in \mathcal{P}$ and $H \in \mathcal{P}$ implies $G \cup H \in \mathcal{P}$.

Some well-known hereditary and additive graph properties are (see [1]): $\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$, $\mathcal{O}_k = \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k + 1 \text{ vertices}\}$ and $\mathcal{D}_k = \{G \in \mathcal{I} : \delta(H) \leq k \text{ for each } H \subseteq G\}$, where $\delta(H)$ is the minimum degree of H .

Let $r, s \in \mathbb{N}$, $r \geq s$, and \mathcal{P} and \mathcal{Q} be two additive and hereditary graph properties. A $(\mathcal{P}, \mathcal{Q})$ -total independent set $T = V_T \cup E_T \subseteq V \cup E$ of a graph G is the union of a set V_T of vertices and a set E_T of edges of G such that for the graphs induced by the sets V_T and E_T it holds that $G[V_T] \in \mathcal{P}$, $G[E_T] \in \mathcal{Q}$, and $G[V_T]$ and $G[E_T]$ are disjoint. Let $\mathcal{T}_{\mathcal{P}, \mathcal{Q}}$ be the set of all $(\mathcal{P}, \mathcal{Q})$ -total independent sets of G .

An (r, s) -fractional $(\mathcal{P}, \mathcal{Q})$ -total coloring of a graph $G = (V, E)$ is a coloring of all $x \in V \cup E$ by s -elements subsets $C(x) \subseteq \{1, \dots, r\}$ such that for each color i , $1 \leq i \leq r$, the set $T_i := \{x \in V \cup E : i \in C(x)\}$ belongs to $\mathcal{T}_{\mathcal{P}, \mathcal{Q}}$. The fractional $(\mathcal{P}, \mathcal{Q})$ -total chromatic number $\chi''_{f, \mathcal{P}, \mathcal{Q}}(G)$ of G is defined as the infimum of all ratios r/s such that G has an (r, s) -fractional $(\mathcal{P}, \mathcal{Q})$ -total coloring.



a $(10, 3)$ -fractional $(\mathcal{D}_1, \mathcal{D}_1)$ -total coloring of K_4

Now, let $L(x)$ be a set of admissible colors for every element $x \in V \cup E$. The graph G is called (a, b) -list $(\mathcal{P}, \mathcal{Q})$ -total colorable if for each list assignment L with $|L(x)| = a$ for all $x \in V \cup E$ it is possible to choose a subset $C(x) \subseteq L(x)$ with $|C(x)| = b$ for all $x \in V \cup E$ such that $T_i := \{x \in V \cup E : i \in C(x)\}$ belongs to $\mathcal{T}_{\mathcal{P}, \mathcal{Q}}$ for every color i . The $(\mathcal{P}, \mathcal{Q})$ -choice ratio $\text{chr}_{\mathcal{P}, \mathcal{Q}}(G)$ of G is defined as the infimum of all ratios a/b such that G is (a, b) -list $(\mathcal{P}, \mathcal{Q})$ -total colorable.

In [4] we prove $\chi''_{f, \mathcal{P}, \mathcal{Q}}(G) = \text{chr}_{\mathcal{P}, \mathcal{Q}}(G)$ for all simple graphs G .

In the talk we give an equivalent definition for the fractional $(\mathcal{P}, \mathcal{Q})$ -total chromatic number $\chi''_{f, \mathcal{P}, \mathcal{Q}}(G)$ and use that definition to construct bounds for the $(\mathcal{P}, \mathcal{Q})$ -total chromatic number for some properties \mathcal{P} and \mathcal{Q} . These bounds are also valid for the $(\mathcal{P}, \mathcal{Q})$ -choice ratio because of the mentioned result.

For example, with $\mathcal{S} = \{H \in \mathcal{I} : H \text{ consists of stars}\}$ and $\mathcal{L} = \{H \in \mathcal{I} : H \text{ consists of paths}\}$ we obtain in [4]:

Theorem 1 *If $\mathcal{P} \supseteq \mathcal{O}_1$, and $K_n \notin \mathcal{P}$, and $\mathcal{S} \subseteq \mathcal{Q} \subseteq \mathcal{D}_1$ or $\mathcal{L} \subseteq \mathcal{Q} \subseteq \mathcal{D}_1$, and $n \geq 3$ then*

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n(n+1)}{2(n-1)}.$$

Theorem 2 *Let C_n be a cycle on $n \geq 3$ vertices. Then*

$$\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(C_n) = \frac{2n}{n-1} = 2 + \frac{2}{n-1}.$$

We will give some more recent results and discuss interesting problems related to this topic.

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Efficient domination in vertex-transitive cubic graphs

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EXTENDED ABSTRACT

Let G be a graph, $G = (V(G), E(G))$, and let $S \subseteq V(G)$. We say that S dominates G if every vertex $x \in V(G) \setminus S$ has a neighbour in S . The set S dominates G efficiently if every vertex $x \in V(G)$ is either in S or has precisely one neighbour in S . The existence of an efficient dominating set can be used to determine the dominance number, i.e., the minimum size of a dominance set. For instance, if G is a regular graph of degree r which admits an efficient dominating set, then its dominance number is $|V(G)|/(r+1)$.

Möbius ladder M_n is a cubic graph obtained from the cycle on $2n$ vertices by adding a perfect matching connecting pairs of opposite vertices. Using an algebraic approach based on lifts we prove that, a connected cubic graph G on 2^m vertices does not admit efficient dominating set if and only if $m \geq 3$ and G is isomorphic to the Möbius ladder $M_{2^{m-1}}$, [1].

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Generalized line graphs: Cartesian products and complexity of recognition

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EXTENDED ABSTRACT

The well-known line graph operator maps graph G to the graph $L(G)$ whose vertices represent the edges (K_2 -subgraphs) of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges share a vertex (a K_1 -subgraph) in G . The more general notion of k -line graphs was also introduced and studied (see e.g., [7, 4]). For a positive integer k , the k -line graph $L_k(G)$ of G has vertices corresponding to the K_k -subgraphs of G and two vertices in $L_k(G)$ are adjacent if and only if the K_k -subgraphs represented share exactly $k-1$ vertices. The 1-line graph of a graph of order n is exactly the complete graph K_n ; 2-line graph means line graph in the usual sense; a 3-line graph is also called "triangle graph".

The structural and algorithmic properties of k -line graphs have been studied recently by the present authors in [3], from where Theorems 1, 2 and 3 below are quoted.

First, answering a question of Bagga [4], we characterize graphs G and F whose Cartesian product is a triangle graph:

Theorem 1 *For two non-edgeless graphs G and F , the Cartesian product $G \square F$ is a triangle graph if and only if G is a complete graph and F is the line graph of a K_3 -free graph (or vice versa).*

For $k \geq 4$, the product of two non-complete graphs can also yield a k -line graph, for instance the grid graph $P_n \square P_m$ is a k -line graph for every n, m and $k \geq 4$. Hence, the condition that at least one of G and F is complete is not necessary. The following generalization of Theorem 1 gives a necessary and sufficient condition for every k .

Theorem 2 *For every $k \geq 2$, the product $G \square F$ of two non-edgeless connected graphs is a k -line graph if and only if there exist positive integers k_1 and k_2 such that G is the k_1 -line graph of a K_{k_1+1} -free graph, F is the k_2 -line graph of a K_{k_2+1} -free graph and $k_1 + k_2 \leq k$ holds.*

Another issue is the recognition problem of k -line graphs. That is, a positive integer k is fixed, and an instance graph G of the problem is given. We have to decide whether there exists a graph G' whose k -line graph is isomorphic to G . For $k = 1$ the recognition is trivial. For $k = 2$, as well-known, line graphs can be recognized in polynomial time, based on the forbidden subgraph characterization of Beineke [5]. Also linear-time algorithms were designed for solving this problem [6, 8]. In sharp contrast to these results, we have proved that the recognition problem of k -line graphs is NP-complete for each fixed $k \geq 3$.

Theorem 3 *The following problems are NP-complete for every fixed $k \geq 3$:*

- (i) *Recognizing k -line graphs.*
- (ii) *Deciding whether a given graph is the k -line graph of a K_{k+1} -free graph.*

Moreover, both problems remain NP-complete on the class of connected graphs.

NP-hardness for $k = 3$ is reduced from that of recognizing “triangular line graphs” (usually termed Anti-Gallai graphs) recently proved by Anand et al. [1]. Then, for larger values of k , we apply Theorem 2 in our reduction.

It is worth noting that there is a further jump between the behavior of 2- and 3-line graphs, which likely is in connection with the jump concerning time complexity. Namely, this further difference is in the uniqueness of preimages. As a matter of fact, each connected 2-line graph different from K_3 has a unique preimage if we disregard isolated vertices. In other words, viewing the situation from the side of preimages, the line graphs of two non-isomorphic graphs containing no isolated vertices and no K_3 -components surely are non-isomorphic. The similar statement is not true for 3-line graphs, even if we suppose that every edge and every vertex of the preimage is contained in a triangle. For example, there are five non-isomorphic graphs whose triangle graph is the 8-cycle. Additionally, the number of essentially different preimages of an n -cycle goes to infinity as $n \rightarrow \infty$ [2].

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Enumerating minimum feedback vertex sets in directed graphs

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EXTENDED ABSTRACT

The minimum feedback vertex set (MFVS) problem for directed graph is one of those considered by Karp in his famous paper of 1972 [4]. Given a directed graph $G = (V, E)$, recall that a *feedback vertex set* is a subset S of V such that each circuit of G is covered by S . In other words, S is a subset of vertices such that the subgraph G' of G induced by $V - S$ is acyclic. The MFVS problem is known to be NP-hard for general classes of graphs [4], even in its versions for undirected graphs or for sets of arcs instead of vertices. It has been studied extensively and has numerous applications in electronics, computers-aided design, deadlock prevention, programming verification and Bayesian inference [1]. Recently, we have shown that it is a graph-theoretical formulation of a fundamental linguistic problem called *symbol grounding problem* [2], which consists in finding minimum sets of words in a dictionary such that we may learn all remaining words starting from these initial sets. It is well-known that the number of minimum feedback vertex sets may be exponential in the size of the graph [3]. Therefore, one might wonder whether efficient data structures exist for storing all solutions in the case of particular classes of graphs.

In [1, 3], the authors consider the problem of finding one MFVS. For example, Lin and Jou introduced the concept of *contractible graphs*, that reduce to the empty graph under the application of eight contraction operators [1]. In particular, one can compute a MFVS in polynomial time whenever the initial graph is contractible and otherwise, they suggest a branch-and-bound algorithm based on their reduction operators. On the other hand, Formin et al. propose another algorithm to compute a MFVS for undirected graphs in $O(1.7548^n)$ by transforming the problem into that of finding maximum induced trees [3]. Finally, in [5], Formin et al. provide an algorithm for enumerating all minimal (inclusion-wise) feedback vertex sets with polynomial delay. To our knowledge, this is the only known non trivial algorithm related to the enumeration of MFVS's.

In this short communication, we propose an algorithm that enumerates all MFVS's (cardinality-wise), as well as a convenient data structure for representing them. As a first step, we restrict our attention on contractible graphs. Our algorithm is based on a branch-and-bound approach as well as the reduction operators of Lin and Jou [1]. Indeed, it turns out that 7 of 8 operators may be naturally extended to keep track of all solutions. More precisely, let $G = (V, E)$ be a directed graph (see Figure 14(a)).

Given $v \in V$, we say that v is *essential* if it belongs to every MFVS of G . Similarly, v is called *useless* if it does not belong to any MFVS. At each step of the algorithm, we efficiently compute essential and useless vertices. On one hand, we remove essential vertices and add them to the current solution. On the other hand, useless vertices are removed while connecting each predecessor with each successor. Next we apply two operators introduced in [1] that act on edges only. When the graph is reduced as much as possible (see Figure 14(b)), we can branch according to any remaining vertex, by including or excluding it from the current partial MFVS. Also, whenever it is possible, we break the problem into smaller parts according to the connected components of the graph, which speeds up significantly the computations.

Since the number of solutions is very large, it is convenient to use a regular expression to represent all the solutions. For instance, a regular expression describing every MFVS of the graph of Figure 14(a) is represented as a tree in Figure 14(c). This representation presents many advantages such as generating efficiently random MFVS's with uniform probability, iterating over all solutions with constant delay, compute the number of solutions in which each vertex appears and verify whether a given set is a MFVS or not.

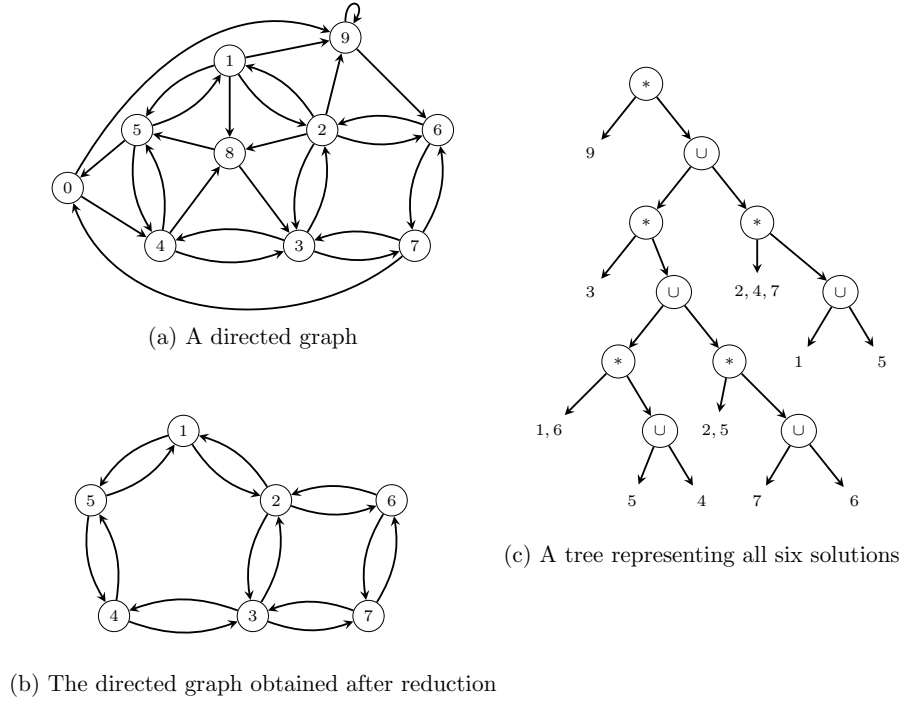


Figure 14: An example. In (c), ‘ \cup ’ represents the union operator and ‘ $*$ ’ represents the concatenation operator, while the leaves are sets of vertices.

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Strong connectivity and directed triangles in oriented graphs. Partial results on a particular case of the Caccetta-Häggkvist conjecture

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EXTENDED ABSTRACT

A particular case of Caccetta-Häggkvist conjecture, says that a digraph of order n with minimum out-degree at least $\frac{1}{3}n$ contains a directed cycle of length at most 3. In a recent paper, Kral, Hladky and Norine (see [2]) proved that a digraph of order n with minimum out-degree at least $0.3465n$ contains a directed cycle of length at most 3 (which currently is the best result). A weaker particular case says that a digraph of order n with minimum semi-degree at least $\frac{1}{3}n$ contains a directed triangle. In a recent paper (see [3]), by using the result of [2], the author proved that for $\beta \geq 0.343545$, any digraph D of order n with minimum semi-degree at least βn contains a directed cycle of length at most 3 (which currently is the best result). This means that for a given integer $d \geq 1$, every digraph with minimum semi-degree d and of order md with $m \leq 2.91082$, contains a directed cycle of length at most 3. In particular, every oriented graph with minimum semi-degree d and of order md with $m \leq 2.91082$, contains a directed triangle. In this paper, by using again the result of [2], we prove that every oriented graph with minimum semi-degree d , of order md with $2.91082 < m \leq 3$ and of strong connectivity at most $0.679d$, contains a directed triangle. This will be implied by a more general and more precise result, valid not only for $2.91082 < m \leq 3$ but also for larger values of m . As application, we improve two existing results. The first result (Authors Broersma and Li in [1]), concerns the number of the directed cycles of length 4 in a possibly triangle free oriented graph of order n and of minimum semi-degree at least $\frac{n}{3}$. The second result (Authors Kelly, Kühn and Osthus in [4]), concerns the diameter of a triangle free oriented graph of order n and of minimum semi-degree at least $\frac{n}{5}$.

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Linear colorings of subcubic graphs

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EXTENDED ABSTRACT

A *proper coloring* of a graph is an assignment of colors to the vertices of the graph so that adjacent vertices receive different colors. Graph coloring is an important topic in graph theory and has wide applications in scheduling and partitioning problems.

Yuster (1998, [13]) introduced the notion of *linear coloring*, which is a proper coloring such that each pair of color classes induce a linear forest, where a linear forest is a union of disjoint paths. This notion combines the well-studied *acyclic colorings* (which are proper colorings so that each pair of color classes induce a forest) introduced by Grünbaum (1973, [6]) and the *frugal colorings* (a proper coloring is *k-frugal* if the subgraph induced by each pair of color classes has maximum degree less than k) introduced by Hind, Molloy, and Reed (1997, [7]).

We write $\text{lc}(G)$ to denote the *linear chromatic number* of G , which is the smallest integer k such that G has a linear coloring with k colors. Yuster [13] constructed an infinite family of graphs such that $\text{lc}(G) \geq C_1 \Delta(G)^{3/2}$, for some constant C_1 . He also proved an upper bound of $\text{lc}(G) \leq C_2 \Delta(G)^{3/2}$, for some constant C_2 and for sufficiently large $\Delta(G)$.

As most coloring problems, it is hard in general to determine the linear chromatic number. For example, Esperet, Montassier, and Raspaud [5] proved that deciding whether a bipartite subcubic graph is linearly 3-colorable is an NP-complete problem. On the other hand, there are some easy upper and lower bounds on $\text{lc}(G)$ for every graph G . Let G be a graph with maximum degree $\Delta(G)$. Then $\text{lc}(G) \geq \lceil \Delta(G)/2 \rceil + 1$, since each color can appear on at most two neighbors of a vertex of maximum degree, and $\text{lc}(G) \leq \chi(G^2) \leq \Delta(G^2) + 1 \leq \Delta(G)^2 + 1$, where $\chi(G)$ denotes the chromatic number of G , and G^2 is the graph obtained from G by adding edges xy for each pair of vertices x, y with distance two. Li, Wang, and Raspaud [9] improved the easy upper bound to $\text{lc}(G) \leq (\Delta(G)^2 + \Delta(G))/2$.

For every family of lists $(L(v) : v \in V(G))$ of size k , we say that a proper coloring f is a *proper L-coloring* of G if $f(v) \in L(v)$ for every vertex v of G . General list-coloring was first introduced by Erdős, Rubin, and Taylor [4] and independently by Vizing [12] in the 1970s, and it has been well-explored since then [8].

The list-version of linear coloring was first studied by Esperet, Montassier, and Raspaud [5]. We say that a proper L -coloring of G is *linear* if the subgraph of G induced by each two color classes is a linear forest. A graph G is *linearly k-choosable* if for every family of lists $(L(v) : v \in V(G))$ of size k , the graph G has a linear L -coloring. When all the lists are the same, it is the same as linear k -coloring. We denote by $\text{lc}_l(G)$ the smallest k so that G is linearly k -choosable.

Clearly, $\text{lc}_l(G) \geq \text{lc}(G)$. Substantial work has been done on the study of graphs whose linear (list) chromatic number is close to the easy lower bound $\lceil \Delta/2 \rceil + 1$, see [1, 2, 3, 5, 9, 10]. On the other hand, a little more is known when a graph has small maximum degree. Li, Wang, and Raspaud [9] showed that $\text{lc}(G) \leq 8$ if $\Delta(G) \leq 4$ and $\text{lc}(G) \leq 14$ if $\Delta(G) \leq 5$. Esperet, Montassier, and Raspaud [5] proved that $\text{lc}_l(G) \leq 9$ if $\Delta(G) \leq 4$ and $\text{lc}_l(G) \leq 5$ if G is subcubic (i.e., $\Delta(G) \leq 3$). Note that $K_{3,3}$ is not linearly 4-colorable, so their upper bound on subcubic graph is tight, but they conjectured $K_{3,3}$ is the only subcubic graph which is not linearly 4-choosable. In this paper, we confirm this conjecture. Note that not only the linearly 4-choosability but also the linearly 4-colorability of general subcubic graphs was not known. As a matter of fact, we prove the following slightly stronger result.

We say that a linear L -coloring is *superlinear* if the neighbors of every vertex of degree two receive different colors. We say that a graph G is *superlinearly k-choosable* if it is

linearly k -choosable in such a way that the corresponding linear coloring can be chosen to be superlinear.

Theorem 1 *Let G be a subcubic graph which has no component isomorphic to $K_{3,3}$ or C_5 . Then G is superlinearly 4-choosable.*

Note that C_5 is linearly 4-choosable but not superlinearly 4-choosable. In addition, our proof of Theorem 4 is constructive and yields a linear-time algorithm to find a superlinear L -coloring when the family of lists $(L(v) : v \in V(G))$ is given. This generalizes an algorithm of Skulrattanakulchai [11] to acyclically color subcubic graphs with four colors.

Now, we sketch the proof of Theorem 1. We say that G is a *minimum counterexample* if it is a subcubic graph without $K_{3,3}$ or C_5 as components and there is a family L of lists of size k such that G has no superlinear L -coloring, but every subcubic graph with fewer vertices than G and with no component isomorphic to $K_{3,3}$ or C_5 is superlinearly 4-choosable. First, we prove that every minimum counterexample is a connected cubic graph which does not contain K_3 or $K_{2,3}$ as an induced subgraph and is not isomorphic to the Petersen graph. Second, we prove that every superlinear L -coloring of $G - C$ can be extended to G , where G is a minimum counterexample and C is a shortest cycle in G . Since the Petersen graph is not a minimum counterexample, the subgraph obtained from any minimum counterexample by deleting a shortest cycle has a superlinear L -coloring for every family of lists of size k . Therefore, no minimum counterexample exists. This finishes the outline of the proof.

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How many matchings of a given size are necessary to cover the edge-set of a graph?

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EXTENDED ABSTRACT

Let m be a positive integer and let G be a graph. An $[m]$ -covering of G is a set $\mathcal{M} = \{M_1, \dots, M_k\}$ of matchings (1-regular subgraphs) of G , each of size m , such that $\cup_{i=1}^k M_i = E(G)$, where $E(G)$ denotes the edge-set of G . We will say a graph $[m]$ -coverable if G admits an $[m]$ -covering. An $[m]$ -covering of smallest order will be called excessive $[m]$ -factorization of G and the order of any excessive $[m]$ -factorization of G will be denoted by $\chi'_{[m]}(G)$ and called excessive $[m]$ -index. We set $\chi'_{[m]}(G) = \infty$ if G is not $[m]$ -coverable.

First we show that the excessive $[m]$ -index is strictly related to outstanding conjectures of Berge, Fulkerson and Seymour and we propose a new conjecture of the same type.

Then we summarize the results obtained for the case in which m is small.

Matchings of large size

An outstanding conjecture of Berge and Fulkerson is usually states as follows: for each bridgeless cubic graph G there exist six perfect matchings of G with the property that each edge of G is contained in exactly two of them. It is straightforward that the Berge-Fulkerson conjecture implies the existence of five perfect matchings covering the edge-set of G ; it is sufficient to select five of the six perfect matchings. I have recently proved in [7] that an equivalent formulation of the Berge-Fulkerson conjecture is the following:

Conjecture 1 *Let G be a bridgeless cubic graph of order $2n$. Then, $\chi'_{[n]}(G) \leq 5$.*

Having in mind this result, we have considered in [2] the case in which G is cubic of order $2n$ and $m = n - 1$, in other words we consider matchings of size one less than a perfect matching. We propose the following conjecture:

Conjecture 2 *Let G be a cyclically 4-connected cubic graph of order $2n$. Then, $\chi'_{[n-1]}(G) = 4$.*

There are some large classes of cubic graphs for which the conjecture is verified to be true: among the others we recall 3-edge-colorable graphs, almost Hamiltonian graphs and graphs having oddness 2 or 4.

Furthermore, we are able to construct an infinite family of 1-connected cubic graphs for which the excessive $[n - 1]$ -index is as large as we want.

In the case of r -regular graphs G of order $2n$ and $r > 3$, the excessive $[n]$ -index cannot be bounded by any constant as proved in [8]. Seymour conjectures in [11] that the “right” condition to put on is about the edge-boundary of odd-sized subsets of the vertex-set. He defines an r -graph as an r -regular graph such that every odd-sized subset of the vertex-set has edge-boundary at least r and for this relevant class of graphs proposes a generalization of the Berge-Fulkerson conjecture. I prove in [9] that also in this case the conjecture can be stated in terms of the excessive $[n]$ -index of G as follows:

Conjecture 3 *Let G be an r -graph of order $2n$. Then, $\chi'_{[n]}(G) \leq 2r - 1$.*

I also exhibit a class of r -graphs for which the bound $2r - 1$ is reached.

Matchings of small size

If m is a small integer the more ambitious task of finding a general formula to compute $\chi'_{[m]}(G)$ can be considered. Obviously $\chi'_{[1]}(G) = |E(G)|$ and it is easy to prove that $\chi'_{[2]}(G) = \max\{\chi'(G), \lceil |E(G)|/2 \rceil\}$ where $\chi'(G)$ denotes the chromatic index of G . The case $m = 3$ is completely solved by Cariolaro and Fu in [3]. We need the following definition to state their result: a set S of edges is a splitting set if no two edges in S belong to the same $[m]$ -matching of G . We denote by $s(G)$ the maximum cardinality of a splitting set of G .

Theorem 4 *Let G be a $[3]$ -coverable graph. Then*

$$\chi'_{[3]}(G) = \max\{\chi'(G), \lceil |E(G)|/3 \rceil, s(G)\}$$

The next step is $m = 4$: we propose in [10] a general formula to compute this index for a tree and we suggest a possible generalization of such a formula to compute the excessive $[4]$ -index of any graph.

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Closures and stability of hereditary classes of graphs

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EXTENDED ABSTRACT

All graphs considered here are simple, finite and undirected. If \mathcal{X} is a family of graphs, we say that a graph G is \mathcal{X} -free if G does not contain any graph from \mathcal{X} as an induced subgraph. The class of all \mathcal{X} -free graphs is denoted $\text{Forb}(\mathcal{X})$. If \mathcal{X} is finite and $\mathcal{X} = \{X_1, \dots, X_k\}$, then we also say that G is (X_1, \dots, X_k) -free and we write $G \in \text{Forb}(X_1, \dots, X_k)$. In this context, the graphs in \mathcal{X} will be referred to as *forbidden induced subgraphs*. Specifically, the graph $C = K_{1,3}$ is called the *claw* and graphs in $\text{Forb}(C)$ are said to be *claw-free*.

For $x \in V(G)$, the *local completion of G at x* is the graph $G_x^* = (V(G), E(G) \cup \{uv \mid u, v \in N_G(x)\})$ (i.e., G_x^* is obtained from G by adding to $\langle N_G(x) \rangle_G$ all missing edges). A vertex $x \in V(G)$ is said to be *eligible* if its neighborhood $N_G(x)$ induces a connected noncomplete graph. The *closure* of a claw-free graph G is the graph $\text{cl}(G)$ obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely, there is a sequence of graphs G_1, \dots, G_k such that $G_1 = G$, $G_{i+1} = (G_i)_x^*$, for some vertex $x \in V(G)$ eligible in G_i , $i = 1, \dots, k-1$, and $G_k = \text{cl}(G)$). In [3], it was proved that, for a claw-free graph G ,

- (i) $\text{cl}(G)$ is uniquely determined,
- (ii) $\text{cl}(G)$ is the line graph of a triangle-free graph,
- (iii) $\text{cl}(G)$ is hamiltonian if and only if G is hamiltonian.

The behavior of some further path and cycle properties under the closure has also been studied. It turns out that the closure operation preserves some of these properties (e.g., the existence of a 2-factor - see [4]), while some other properties (such as e.g. the Hamilton-connectedness, see [1]) are not necessarily preserved. Consequently, several further closure concepts have been developed - see e.g., [2], [5] for strengthening of the closure $\text{cl}(G)$ for hamiltonicity, [6] for a closure for 2-factors, or [7] for a closure for Hamilton-connectedness. All these closure concepts are based on the local completion operation, i.e., the closure of a graph G is obtained by a series of local completions. The difference is in the definition of eligibility.

Let \mathcal{C} be a subclass of the class of claw-free graphs. We say that \mathcal{C} is *stable* if $G \in \mathcal{C}$ implies $G_x^* \in \mathcal{C}$, for any $x \in V(G)$. If cl is any closure operation based on local completions (i.e., such that $\text{cl}(G)$ is obtained from G by a series of local completions), then a class \mathcal{C} being stable implies $\text{cl}(G) \in \mathcal{C}$, for any $G \in \mathcal{C}$.

For various types of closure operations, stable classes have been studied, and for some of them complete characterizations are known; however, proofs of these results are lengthy and technical. In this work, we show a common background for results of this type by identifying the corresponding concept in the line graph preimages.

Let G be a graph, $x \in V(G)$ a vertex of degree at least 2, and let A, B be a partition of $E(x)$ (i.e., $E(x) = A \cup B$, where A, B are disjoint and nonempty). Let G_x^+ be the graph with $V(G_x^+) = (V(G) \setminus \{x\}) \cup \{x_1, x_2\}$, where $x_1, x_2 \notin V(G)$ (i.e., x_1 and x_2 are "new" vertices), in which for every edge $e \in E(G)$ with vertices u, x the graph G_x^+ contains an edge from u to x_1 if $e \in A$, or from u to x_2 if $e \in B$, respectively. We say that G_x^+ is obtained from G by *splitting of type 1 of the vertex x* .

Let $e \in E(G)$ be a pendant edge with vertices u, x , $d_G(x) \geq 3$. We say that a graph $G_x^{+(e)}$ is obtained from G by *splitting of type 2 of x* , if $G_x^{+(e)}$ is obtained from the graph $(V(G) \setminus \{u\}, E(G) \setminus \{e\})$ by splitting of type 1 of x . (see Fig. 1).

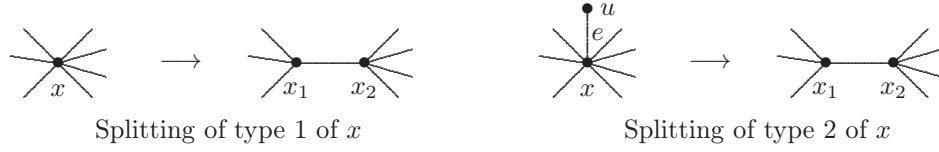


Figure 1: Vertex splitting

Let \mathcal{Y} be a family of graphs. We say that \mathcal{Y} is *closed under vertex splitting* (or briefly *split-closed*), if, for any $Y \in \mathcal{Y}$ and any Y' obtained from Y by vertex splitting (of type 1 or 2), Y' contains a subgraph $Y'' \in \mathcal{Y}$.

The following result characterizes families of closed forbidden subgraphs that yield stable classes (here a *cherry* is a pair of pendant edges with a common vertex, and we write $L(H)$ for the line graph of a graph H).

Theorem 1 *Let \mathcal{H} be the class of all triangle-free and cherry-free graphs and let $\mathcal{Y} \subset \mathcal{H}$ and $\mathcal{X} = L(\mathcal{Y})$. Then the class $\text{Forb}(C, \mathcal{X})$ is stable if and only if \mathcal{Y} is split-closed.*

In case of finite sets \mathcal{X} of forbidden subgraphs, it turns out that two of them are, in a sense, “default”, present in all such sets \mathcal{X} (here, a *generalized net* is the graph obtained by attaching a path to each vertex of a triangle).

Theorem 2 *Let \mathcal{X} be a finite family of line graphs of triangle-free and cherry-free graphs such that $\text{Forb}(C, \mathcal{X})$ is stable. Then \mathcal{X} contains a path or a generalized net.*

In the case of infinite families of forbidden subgraphs an analogue of Theorem 2 is not true, as can be seen e.g. by considering the stable class $\mathcal{X}_k = \text{Forb}(C, \{C_i\}_{i=k}^{\infty})$ (for any fixed $k \geq 3$); \mathcal{X}_k is stable although neither of the forbidden subgraphs is a path or a generalized net. However, it is still possible to show that each family of forbidden subgraphs that yields a stable class contains a proper subfamily with the same property.

Theorem 3 *Let \mathcal{X} be a family of line graphs of triangle-free and cherry-free graphs such that $\text{Forb}(C, \mathcal{X})$ is stable. Then \mathcal{X} contains a subfamily $\mathcal{X}' \subset \mathcal{X}$, $\mathcal{X}' \neq \mathcal{X}$, such that $\text{Forb}(C, \mathcal{X} \setminus \mathcal{X}')$ is stable, unless $|\mathcal{X}| = 1$ and \mathcal{X} contains a path or a generalized net.*

We also discuss the technique of *stabilizers*, which allows to handle unstable classes by proving that their closure can be included into another (possibly stable) class.

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Mapping planar graphs into projective cubes

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EXTENDED ABSTRACT

The projective cube of dimension $2k$, denoted $PC(2k)$, is the graph obtained from the hypercube of dimension $2k + 1$ by identifying each pair of antipodal vertices. With the terminology of Cayley graphs, $PC(2k)$ is $(\mathbb{Z}_2^{2k}, \{e_1, e_2, \dots, e_{2k}, J\})$ where the e_i 's are the vectors of the standard basis and J is the all-1 vector.

A *homomorphism* of a graph G to a graph H is a mapping of $V(G)$ to $V(H)$ which preserves adjacency. We raised the following question:

Problem 1 *Given two integers $r \geq k$, what are the minimal subgraphs of $PC(2k)$ to which every planar graph of odd-girth $2r + 1$ admits a homomorphism to?*

In this talk, we show that this question, surprisingly captures many well-known theorems and conjectures such as the four colour theorem, Grötzsch's theorem, Jeager's conjecture and Seymour's conjecture on edge-coloring of planar graphs. We also show a strong relation between this question and the development of theories such as edge-coloring, fractional coloring, circular coloring for the class of planar graphs. The question is also related to a conjecture on the characterization of binary clutters.

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Matchings in Random Biregular Bipartite Graphs

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EXTENDED ABSTRACT

Let $k \in \mathbb{Q}^+$ be a positive rational number, $n \in \mathbb{Z}^+$ a positive integer that satisfies $kn \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$ a positive integer that satisfies $1 \leq d \leq n$ and $kd \in \mathbb{Z}^+$. Let Y be a set of size n and Z be a set of size kn . Define $\mathcal{G}(k, n, d)$ to be the family of biregular bipartite directed graphs on the vertex set (Y, Z) (with edges directed from Y to Z) where $d^+(y) = kd$ for all $y \in Y$ and $d^-(z) = d$ for all $z \in Z$. A *random biregular bipartite graph* (with parameters k, n, d) is a graph chosen from $\mathcal{G}(k, n, d)$ uniformly at random. The corresponding model of random graphs is denoted by $G(k, n, d)$.

The case where $k = 1$ has a special relevance since $\mathcal{G}(1, n, d)$ is the family of regular bipartite graphs of size n and degree d where the edges are canonically oriented from one stable set to the other. Estimating the size of $\mathcal{G}(1, n, d)$ as a function of d and n is a question that has been studied extensively [2, 4].

On the first part we study the probability of having a perfect matching in an induced subgraph H of $\mathcal{G}(k, n, d)$ and how this probability changes with d .

Theorem 1 *Under the assumptions from above, let $G \sim G(k, n, d)$. Take subsets $A \subseteq Y$ and $B \subseteq Z$ of size kd and define $H := G[A, B]$ to be the subgraph induced by G on vertex set (A, B) . Then*

- (i) *No perfect matching exists in H whp when $\frac{kd^2}{n} - \log(kd) \rightarrow -\infty$ or when d is a constant.*
- (ii) *A perfect matching exists in H whp when $\frac{kd^2}{n} - \log(kd) \rightarrow +\infty$.*

A *random bipartite graph* is a bipartite graph on the vertex set (A, B) where edges are chosen independently of each other with probability p . The model of random bipartite graphs is denoted by $B(n, p)$. The existence of perfect matchings in random bipartite graphs was investigated by Erdős and Rényi about fifty years ago. In [1] they showed that if $G' \sim B(n, p)$ and $p = \frac{\log n + c}{n}$, then

$$\Pr(\text{There exists a perfect matching in } G') = (1 + o(1)) \exp(-2e^{-c}).$$

Theorem 1 is an Erdős–Rényi type result for the induced subgraph H . To make the similarity between Theorem 1 and the result about random bipartite graphs, as clear as possible we set $k = 1$ in the former. The induced subgraph H is somewhat similar to a random bipartite graph as it has similar properties to $G' \sim B(d, d/n)$: The size of the stables of H is d and edges appear in H with uniform probability d/n . The main difference is that edges do not appear independently in H , yet the dependence is generally speaking small. The similarity between H and G' is reflected by the fact that a perfect matching exists in both graphs whp when $d^2/n - \log d \rightarrow +\infty$.

The main motivation to study the existence of perfect matchings in induced subgraphs of random biregular bipartite graphs has to do with commutative graphs and Plünnecke's inequality. A very comprehensive study of the applications of commutative graphs and Plünnecke's inequality can be found in [8]. Here we only present the necessary facts that relate commutative graphs with Theorem 1.

For any $y \in Y$ we denote by $\Gamma(y) = \{z \in Z : yz \in E(G)\}$, and by $\Gamma^{-1}(z)$ the inverse neighbourhood of $z \in Z$.

A directed layered graph G with vertex set $X_0 \cup X_1 \cup \dots \cup X_h$ is called *commutative* if

1. There are edges only between consecutive layers, so that $E(X_i, X_j) = \emptyset$ unless $j = i + 1$ for all $0 \leq i, j \leq h$.

2. For all $1 \leq i \leq h$ and $uv \in E(X_{i-1}, X_i)$ there exists a perfect matching from $\Gamma(u)$ to $\Gamma(v)$.
3. For all $1 \leq i \leq h$ and $uv \in E(X_i, X_{i+1})$ there exists a perfect matching from $\Gamma^{-1}(v)$ to $\Gamma^{-1}(u)$.

Plünnecke introduced commutative graphs to study the growth of sumsets [6, 7]. He was interested in the *magnification ratios* of graphs,

$$D_i(G) = \min_{\emptyset \neq Z \subseteq X_0} \frac{|\Gamma^{(i)}(Z)|}{|Z|},$$

for $1 \leq i \leq h$ and $\Gamma^{(i)}(Z)$ is defined iteratively by $\Gamma^{(i)}(Z) = \Gamma(\Gamma^{(i-1)}(Z))$. Plünnecke proved in [6] that if G is a commutative graph, then the sequence $D_i(G)^{1/i}$ is decreasing. In [5] it was shown that the upper bound for $D_i(G) \leq D_1(G)^i$ is sharp.

We apply a variant of Theorem 1 where $B = \Gamma(y)$ for some $y \in A$, to give a non-constructive, and probabilistic in nature, proof of the existence of graphs that are extremal for Plünnecke's inequality, answering a question of Gowers.

We form a layered directed graph by “placing random biregular bipartite directed graphs on top of each other.” This simple construction works when the out-degree is large enough compared to the size of the bottom layer and the resulting graph is *whp* commutative.

Theorem 2 *Let X_0, X_1, \dots, X_h be sets with $|X_i| = k^i m$. Suppose that $d \geq 2$ and for $1 \leq i \leq h$ let $G_i := G[X_{i-1}, X_i] \sim G(k, k^{i-1}m, d)$.*

Let G be a graph with $V(G) = X_0 \cup \dots \cup X_h$ and $E(G) = \cup_{i=1}^h E(G_i)$. Then

- (i) *The graph G is not commutative whp when $d \leq \sqrt{\frac{1}{3}k^{h-2}m \log(km)}$.*
- (ii) *The graph G is commutative whp when $d \geq 3\sqrt{k^{h-2}m \log(km)}$.*

The methods we use in the paper are somewhat different to those that appear in the literature. Our strategy is to mirror the proof for random bipartite graphs of Erdős and Rényi. The biggest obstacle is dealing with dependencies among the edges. We do this by repeatedly using three ingredients: the regularity of the degrees, the symmetry of $\mathcal{G}(k, n, d)$ and the idea of switching edges in graphs introduced by McKay in [3].

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Proper edge colourings distinguishing colour pallets of the neighbours - algorithms enhanced by Combinatorial Nullstellensatz

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EXTENDED ABSTRACT

Consider a simple graph $G = (V, E)$ and its *proper* edge colouring c with the elements of the set $\{1, 2, \dots, k\}$. We say that c is *neighbour set distinguishing* (or *adjacent strong*) if for every edge $uv \in E$, the set of colours incident with u is distinct from the set of colours incident with v . Let us then consider a stronger requirement and suppose we wish to distinguishing adjacent vertices by sums of their incident colours. In both problems the challenging conjectures presume that such colourings exist for any graph G containing no isolated edges if only $k \geq \Delta(G) + 2$. We prove that in both problems $k = \Delta(G) + 3\text{col}(G) - 4$, see [3] (or $k = 2\Delta(G) + \text{col}(G) - 1$, see [2]) is sufficient. The proofs are based on the Combinatorial Nullstellensatz, applied in the “sum environment”.

Theorem 1 (Alon [1], Combinatorial Nullstellensatz) *Let \mathbb{F} be an arbitrary field, and let $P = P(x_1, \dots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$. Suppose the degree $\deg(P)$ of P equals $\sum_{i=1}^n k_i$, where each k_i is a non-negative integer, and suppose the coefficient of $x_1^{k_1} \dots x_n^{k_n}$ in P is non-zero. Then if S_1, \dots, S_n are subsets of \mathbb{F} with $|S_i| > k_i$, there are $s_1 \in S_1, \dots, s_n \in S_n$ so that $P(s_1, \dots, s_n) \neq 0$.*

The following result of Scheim is also crucial for the algebraic part of our approach.

Theorem 2 (Scheim [4]) *If $P(x_1, x_2, \dots, x_n) \in \mathbb{R}[x_1, x_2, \dots, x_n]$ is of degree $\leq s_1 + s_2 + \dots + s_n$, where s_1, s_2, \dots, s_n are nonnegative integers, then*

$$\begin{aligned} & \left(\frac{\partial}{\partial x_1} \right)^{s_1} \left(\frac{\partial}{\partial x_2} \right)^{s_2} \dots \left(\frac{\partial}{\partial x_n} \right)^{s_n} P(x_1, x_2, \dots, x_n) \\ &= \sum_{x_1=0}^{s_1} \dots \sum_{x_n=0}^{s_n} (-1)^{s_1+x_1} \binom{s_1}{x_1} \dots (-1)^{s_n+x_n} \binom{s_n}{x_n} P(x_1, x_2, \dots, x_n). \end{aligned}$$

In fact the identical upper bounds for the size of the colour set (i.e., $k \leq \Delta(G) + 3\text{col}(G) - 4$ and $k \leq 2\Delta(G) + \text{col}(G) - 1$) also hold if we use any set of k real numbers instead of $\{1, 2, \dots, k\}$ as edge colours, and the same is true in list versions of the both concepts. In particular, we therefore obtain that lists of length $\Delta(G) + 14$ ($\Delta(G) + 13$ in fact) are sufficient for planar graphs.

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Largest planar ocliques

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EXTENDED ABSTRACT

An *oriented graph* \vec{G} is a directed graph obtained by replacing each edge uv of a simple graph G with an arc (ordered pair of vertices) $u\vec{v}$ or $v\vec{u}$. The graph \vec{G} is an *orientation* of G and G is the *underlying graph* of \vec{G} , denoted by $\text{und}(\vec{G})$. We denote by $V(\vec{G})$ and $A(\vec{G})$ respectively the set of vertices and arcs of \vec{G} . Similarly, $V(G)$ and $E(G)$ denote respectively the set of vertices and edges of G . The set of all vertices adjacent to a vertex v in a graph is the set of *neighbours* and is denoted by $N(v)$. For oriented graphs, if there is an arc $u\vec{v}$, then u is an *in-neighbour* of v and v is an *out-neighbour* of u . The sets of all in-neighbours and out-neighbours of v are denoted by $N^-(v)$ and $N^+(v)$ respectively. A path obtained by two consecutive arcs $u\vec{v}$ and $v\vec{w}$ is called a *2-dipath*. In this article, by graph we will mean either a simple undirected graph or an oriented graph.

An *oriented k -colouring* [1] of an oriented graph \vec{G} is a mapping f from the vertex set $V(\vec{G})$ to the set $\{1, 2, \dots, k\}$ such that, (i) $f(u) \neq f(v)$ whenever u and v are adjacent and (ii) if $x\vec{y}$ and $u\vec{v}$ are two arcs in \vec{G} , then $f(x) = f(v)$ implies $f(y) \neq f(u)$. The *oriented chromatic number* $\chi_o(\vec{G})$ of an oriented graph \vec{G} is the smallest integer k for which \vec{G} has an oriented k -colouring. An *oriented clique* or simply *oclique* is an oriented graph whose any two distinct vertices are either adjacent or connected by a 2-dipath. Ocliques are therefore precisely those oriented graphs \vec{G} for which $\chi_o(\vec{G}) = |\vec{G}| = |V(\vec{G})|$. Note that an oriented graph with an oclique of order n as a subgraph has oriented chromatic number at least n , where the *order* of a graph G is the number of its vertices, denoted by $|G|$.

For the family \mathcal{P} of planar graphs we have $17 \leq \chi_o(\mathcal{P}) \leq 80$ where the lower bound is due to Marshall [4] and the upper bound to Raspaud and Sopena [2]. Tightening these bounds are challenging problems in the domain of oriented colouring. A naturally related question to this problem is: What is the maximum order of a planar oclique? In order to find the answer to this question, Sopena [5] found a planar oclique of order 15 while Klostermeyer and MacGillivray [6] showed that there is no planar oclique of order more than 36 and conjectured that the maximum order of a planar oclique is 15. The similar question for planar ocliques with given girth, where the *girth* of a graph is the length (number of edges) of the smallest cycle in the graph is also of interest and was asked by Klostermeyer and MacGillivray [6]. In this paper we manage to answer these questions and also settle the conjecture by proving the following result:

Theorem 1 (a) If \vec{G} is a planar oclique, then $|\vec{G}| \leq 15$.

(b) If \vec{G} is a planar oclique with girth at least 4, then $|\vec{G}| \leq 6$.

(c) If \vec{G} is a planar oclique with girth at least 5, then $|\vec{G}| \leq 5$.

(d) If \vec{G} is a planar oclique with girth at least k , where $k \geq 6$, then $|\vec{G}| \leq 3$.

The *distance* $d(u, v)$ between two vertices u, v of a graph G is the length (number of edges or arcs) of a shortest path joining u and v . The *diameter* of a graph G is the maximum of $d(u, v)$ taken over all $(u, v) \in V(G) \times V(G)$. Clearly any oclique has diameter at most 2.

For a graph G , $D \subseteq V(G)$ *dominates* G if any vertex of G is either in D or adjacent to a vertex in D . The *domination number* of a graph G is the minimum cardinality of a dominating set. Goddard and Henning [3] showed that every planar graph of diameter 2 has domination number at most 2 except for a particular graph on eleven vertices.

Let \vec{B} be a planar oclique dominated by the vertex v . Sopena [1] showed that any oriented outerplanar graph has an oriented 7-colouring. Hence let c be an oriented 7-colouring of the oriented outerplanar graph obtained from \vec{B} by deleting the vertex v . Now for $u \in N^\alpha(v)$

let us assign the colour $(c(u), \alpha)$ to u for $\alpha \in \{+, -\}$ and the colour 0 to v . It is easy to check that this is an oriented 15-colouring of \vec{B} . Hence any planar oclique dominated by one vertex has order at most 15.

Idea of the proof of Theorem 4: Let \vec{G} be a planar oclique with $|\vec{G}| > 15$. Clearly \vec{G} has diameter 2. Then by the above discussion, the domination number of \vec{G} is 2. Without loss of generality, we may assume that \vec{G} is triangulated.

We define the partial order \preceq for the set of all dominating sets of order 2 of \vec{G} as follows: for any two dominating sets $D = \{x, y\}$ and $D' = \{x', y'\}$ of order 2 of \vec{G} , $D' \preceq D$ if and only if $|N(x') \cap N(y')| \leq |N(x) \cap N(y)|$.

Let $D = \{x, y\}$ be a maximal (with respect to \preceq) dominating set of order 2 of \vec{G} with $C = N(x) \cap N(y)$.

We will restrict the possible values of $|C|$ for which \vec{G} can be a planar oclique of order at least 16. In particular, we will show that $|C| \geq 5$ and $|C| \leq 6$ to force a contradiction. This will prove Theorem 4(a).

Other parts of the Theorem will follow using the classification of planar graphs with girth at least 4 and diameter 2 done by Plesník (1975). \square

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Lovász ϑ function and products of graphs

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EXTENDED ABSTRACT

Hedetniemi conjectured in 1966 that the chromatic number of a categorical product of two graphs is equal to the minimum of the chromatic numbers of the two factors. Despite many efforts, this conjecture is still open. We prove a variant of this where chromatic number is replaced by a strict vector chromatic number, better known as $\vartheta(\bar{G})$, the Lovász' theta function of the complement.

Graph homomorphisms and products This is a paper about relation of graph homomorphism, graph products, and certain graph parameters defined by semidefinite programs. We start by defining the required notions.

Let G, H be graphs (by which we mean undirected simple graphs). We call a mapping $f : V(G) \rightarrow V(H)$ a *graph homomorphism* if for every edge uv of G the graph H contains the edge $f(u)f(v)$. We denote the existence of such homomorphism by $G \rightarrow H$. Many graph theoretic notions have simple expression in terms of homomorphisms, in particular a graph G is k -colorable if and only if $G \rightarrow K_k$.

Let G, H be graphs. We define two graphs with vertex set $V(G) \times V(H)$. In the *categorical product* $G \times H$ tuples (u_1, v_1) and (u_2, v_2) are connected by an edge if and only if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

In the *Cartesian product* $G \square H$ tuples (u_1, v_1) and (u_2, v_2) are connected by an edge if and only if

$$(u_1u_2 \in E(G) \text{ and } v_1 = v_2) \text{ or } (u_1 = u_2 \text{ and } v_1v_2 \in E(H)).$$

It is easy to see that $G \rightarrow G \square H$ (and also $H \rightarrow G \square H$). Indeed, $G \square H$ contains copies of both G and H . A moment's thought reveals that we have $G \times H \rightarrow G$ and $G \times H \rightarrow H$. (This is indeed satisfied in any category and $G \times H$ is called categorical product because it is, in fact, a product in the sense of category theory.) It follows from the above that

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\} \quad (3)$$

and

$$\chi(G \square H) \geq \max\{\chi(G), \chi(H)\}. \quad (4)$$

It is well-known (and simple to prove) that equality holds in (4). Deciding, whether equality holds in (3) is much harder. However, it is motivated by more reasons than by a desire to understand the graph product. It was shown that the Burr-Lovász conjecture from Ramsey theory follows from the following one, conjectured by Hedetniemi [1].

Conjecture 1 (Hedetniemi)

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}$$

Lovász' $\bar{\vartheta}$ -function The so-called Lovász ϑ -function (to be defined shortly) started its existence in [4] as a means to study Shannon capacity of communication channel. It proved its importance in other ways since. In particular, recent development starting with [2] uses ϑ (and its variants, such as so-called vector chromatic number) to approximate chromatic number of graphs.

One important feature, that will not appear directly in this paper, is that ϑ can be defined using semidefinite programming. This perhaps explains (by means of semidefinite duality) why there are so many equivalent definitions. Indeed, a very nice introduction to the topic [3] lists five equivalent formulations. Of these, we will need three.

Definition 2 Let G be a graph. A mapping $a : V(G) \rightarrow \mathbb{R}^d$ (for any integer d) is called an orthogonal labeling of G iff $a(u) \cdot a(v) = 0$ whenever $uv \notin E(G)$.

A mapping $s : V(G) \rightarrow \mathbb{R}^d$ is called strict vector k -coloring if for every vertex v we have $\|s(v)\| = 1$ and $s(u) \cdot s(v) = -\frac{1}{k-1}$ whenever $uv \in E(G)$.

For a vector $x \in \mathbb{R}^d$ we define its cost as $c(x) = x_1^2 / \|x\|^2$.

Next, we introduce the definition of the Lovász' $\vartheta(G)$ function, or, rather, the function $\bar{\vartheta}(G) = \vartheta(\bar{G})$ (where \bar{G} is the complement of a graph G), as $\bar{\vartheta}$ is the topic of this paper.

Definition 3 Let G be a graph. We define $\bar{\vartheta}(G)$ to be the value of any of the following optimization programs:

$$\min \left\{ \frac{1}{t} : \text{there is an OL } a \text{ of } \bar{G} \text{ s.t. } \forall v : c(a_v) = t \right\} \quad (5)$$

$$\min \{k : \text{there is a strict } k\text{-coloring of } G\} \quad (6)$$

$$\max \left\{ \sum_v c(b_v) : b \text{ is an OL of } G \right\} \quad (7)$$

The following lemma was implicit in many papers about strict vector coloring, it did not appear to be much appreciated, though.

Proposition 4 Suppose $G \rightarrow H$ (i.e., there is a homomorphism from G to H). Then

$$\bar{\vartheta}(G) \leq \bar{\vartheta}(H).$$

Results In this paper we start our pursuit to study the SDP-type chromatic parameters in the spirit of chromatic theory of graphs. Namely, we prove a version of Conjecture 1 for $\bar{\vartheta}$. Perhaps surprisingly, we need to find the value of $\bar{\vartheta}(G \square H)$ first. So, the following is our main result.

Theorem 5

$$\begin{aligned} \bar{\vartheta}(G \square H) &= \max\{\bar{\vartheta}(G), \bar{\vartheta}(H)\} \\ \bar{\vartheta}(G \times H) &= \min\{\bar{\vartheta}(G), \bar{\vartheta}(H)\} \end{aligned}$$

In this place we want to mention a recent result of Zhu [5] that proves the version of Hedetniemi conjecture for χ_f . It is also shown there, that the Burr-Lovász conjecture does follow from this version. Similarly to that proof, our present proof relies on the fact, that one can express the studied parameter as a maximum and as a minimum (by means of duality). It seems that it is the lack of such characterization for χ that makes the original Hedetniemi conjecture harder.

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H-kernels in infinite digraphs

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EXTENDED ABSTRACT

Let H be a digraph possibly with loops and D a digraph (possibly infinite) without loops whose arcs are coloured with the vertices of H (D is an H -coloured digraph). A directed walk or a directed path W in D is an H -walk or an H -path, respectively, if and only if the consecutive colors encountered on W form a directed walk in H . A set $N \subseteq V(D)$ is an H -kernel if no two vertices of N have an H -path between them and any $u \in V(D) \setminus N$ reaches some $v \in N$ on an H -path.

Let D be an arc-coloured digraph, in [3] Galeana-Sánchez introduced the concept of color-class digraph of D , denoted by $\mathcal{C}_C(D)$, as follows: vertex set of the color-class digraph are the colors represented in the arcs of D and $(i, j) \in A(\mathcal{C}_C(D))$ if and only if there exist two arcs namely (u, v) and (v, w) in D such that (u, v) has color i and (v, w) has color j .

Since $V(\mathcal{C}_C(D)) \subseteq V(H)$, the main question is: What structural properties of $\mathcal{C}_C(D)$, with respect to H , imply that D has an H -kernel?

In this talk we are going to show conditions on the color-class digraph that guarantee the existence of an H -kernel in infinite digraphs.

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Hunting for rainbow bulls and diamonds

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EXTENDED ABSTRACT

We use [1] for terminology and notation not defined here and consider finite and simple graphs only. If K_n is edge-coloured in a given way and a subgraph H contains no two edges of the same colour, then H will be called a totally multicoloured (TMC) or rainbow subgraph of K_n and we shall say that K_n contains a TMC or rainbow H . For a graph H and an integer n , let $f(n, H)$ denote the maximum number of colours in an edge-colouring of K_n with no TMC H . The numbers $f(n, H)$ are called *anti-ramsey numbers* and have been introduced by Erdős, Simonovits and Sós [2].

We now define $rb(n, H)$ as the minimum number of colours such that any edge-colouring of K_n with at least $rb(n, H) = f(n, H) + 1$ colours contains a TMC or rainbow subgraph isomorphic to H . The numbers $rb(n, H)$ will be called *rainbow numbers*.

For a given family \mathcal{H} of finite graphs $ext(n, \mathcal{H}) =: \max\{|E(G)| \mid H \not\subseteq G \text{ if } H \in \mathcal{H}\}$, that is, let $ext(n, \mathcal{H})$ be the maximum number of edges a graph G of order n can have if it has no subgraph from \mathcal{H} . The graphs attaining the maximum for a given n are called extremal graphs. The numbers $ext(n, \mathcal{H})$ are called *Turán numbers* [9].

For a given graph H , let \mathcal{H} be the family of all graphs which are obtained by deleting one edge from H . If G is a graph of order n having no subgraph isomorphic to H , then a TMC copy of G and one extra colour for all remaining edges (of K_n) has no TMC subgraph H . Hence, $f(n, H) \geq ext(n, \mathcal{H}) + 1$. Moreover,

$$ext(n, \mathcal{H}) + 2 \leq f(n, H) + 1 = rb(n, H) \leq ext(n, H).$$

The lower bound is sharp for some graph classes. This has been shown if H is a complete graph on $k \geq 3$ vertices in [6, 8] and if H is a matching with k edges and $n \geq 2k + 1$ in [3, 8].

Erdős, Simonovits and Sós [2] showed that $f(n, H)/\binom{n}{2} \rightarrow 1 - \frac{1}{d}$ as $n \rightarrow \infty$, where $d+1 = \min\{\chi(H-e) \mid e \in E(H)\}$, and that $f(n, H) - ext(n, \mathcal{H}) = o(n^2)$. Hence the rainbow numbers are asymptotically known if $\min\{\chi(H-e) \mid e \in E(H)\} \geq 3$. If $\min\{\chi(H-e) \mid e \in E(H)\} \leq 2$, then the situation is quite different.

For cycles the following result (which has been conjectured by Erdős, Simonovits and Sós [2]) has been shown by Montellano-Ballesteros and Neumann-Lara [7].

Theorem 1 [7] *Let $n \geq k \geq 3$. Then $rb(n, C_k) = \lfloor \frac{n}{k-1} \rfloor \binom{k-1}{2} + \binom{r}{2} + \lceil \frac{n}{k-1} \rceil$, where r is the residue of n modulo $k-1$.*

Gorgol [4] has considered a cycle C_k with a pendant edge, denoted C_k^+ , and computed all rainbow numbers.

Theorem 2 [4] *$rb(n, C_k^+) = rb(n, C_k)$, for $n \geq k + 1$.*

However, if we add two (or more) edges to a cycle C_k , the situation becomes surprisingly interesting.

Theorem 3 *Let F be a graph of order $n \geq k \geq 3$ containing a cycle C_k . If F has cyclomatic number $v(F) \geq 2$, then $rb(n, F)$ has no upper bound which is linear in n .*

We first consider the graph $D = K_4 - e$, which is called the *diamond*. This graph contains a C_3 and has cyclomatic number $v(D) = 2$. Montellano-Ballesteros [5] has shown an upper bound for the rainbow number of the diamond.

Theorem 4 [5] For every $n \geq 4$,

$$\text{ext}(n, \{C_3, C_4\}) + 2 \leq \text{rb}(n, D) \leq \text{ext}(n, \{C_3, C_4\}) + (n + 1).$$

Using this we can show the following theorem.

Theorem 5 $\text{rb}(n, D) = \Theta(n^{\frac{3}{2}})$.

If $v(F) = 1$, then the situation is quite different. Let B be the unique graph with 5 vertices and degree sequence $(1, 1, 2, 3, 3)$, which is called the *bull*. Here we have been able to compute all rainbow numbers for the bull.

Theorem 6 $\text{rb}(5, B) = 6$ and $\text{rb}(n, B) = n + 2$ for $n \geq 6$.

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Finding Quasi-Hamiltonian Paths and Cycles in Semicomplete Multipartite Digraphs

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EXTENDED ABSTRACT

A digraph $D = (V, A)$ is **semicomplete multipartite** if whenever there is an arc between u and v but no arc between u and w , there is an arc between v and w . Equivalently, there is a partition V_1, V_2, \dots, V_c of V into independent sets so that every vertex in V_i is adjacent to every vertex in V_j for $i \neq j$. In other words semicomplete multipartite digraphs (SMDs) are a generalization of multipartite tournaments where we allow 2-cycles. A **semicomplete digraph** is a SMD where the size of a largest independent set is one.

In 1980 Thomassen [1] gave a complete characterization of weakly hamiltonian connected tournaments (that is there is a hamiltonian path connecting x and y for all vertex pairs $x, y \in D$). From this characterization one can derive a polynomial algorithm for finding the longest $[x, y]$ -path in a semicomplete digraph.

In 1992 in [2] Bang-Jensen, Manoussakis and Thomassen gave a polynomial algorithm for testing whether a given semicomplete digraph contains an (x, y) -hamiltonian path (that is a hamiltonian path from x to y). Interestingly, the algorithm cannot be modified to solve the problem of finding the longest (x, y) -path in a semicomplete digraph and the complexity of this problem is still open.

Conjecture 1 [3] 1998

There exists a polynomial algorithm for finding the longest (x, y) -path in a semicomplete digraph.

For multipartite tournaments it is an open problem whether the existence of a hamiltonian path between x and y can be decided in polynomial time and the problem is most likely quite difficult. For instance the hamiltonian cycle problem is known to be polynomially solvable for semicomplete multipartite digraphs [4], but the algorithm is highly non-trivial. This makes it interesting to study problems for multipartite tournaments which contain the abovementioned hamiltonian path problems for tournaments as special cases. We investigated problems related to the following generalization of hamiltonian paths (cycles) in tournaments.

A **quasi-hamiltonian** path (cycle) in a semicomplete multipartite digraph D is a path (cycle) which visits each maximal independent set (also called a partite set) of D at least once.

Theorem 2 *It is NP-complete to decide whether a semicomplete multipartite digraph contains an (x, y) -quasi-hamiltonian path.*

For the subclass of extended semicomplete digraphs (which are obtained from semicomplete digraphs by substituting independent sets for vertices) a polynomial algorithm does exist by reducing the problem to the (x, y) -hamiltonian path problem for semicomplete digraphs.

Guo, Lu and Surmacs [5] gave a characterization of weakly quasi-hamiltonian-set-connected multipartite tournaments thus generalizing a well-known result for tournaments by Thomassen [1].

The main part of our work was done on weakly quasi-hamiltonicity in SMDs, that is we investigated the complexity of the following problem.

Problem 3 *Decide if a given semicomplete multipartite digraph with vertices x and y contains a quasi-hamiltonian path connecting x and y .*

This further generalizes the problem solved by Thomassen [1]. To get a better understanding of the structure of SMDs we started by introducing some lemmas, the first of which is a generalization of the merging lemma from [2].

Lemma 4 *Let D be a semicomplete multipartite digraph with vertices s_1, s_2, t_1 and t_2 . Then any pair of an (s_1, t_1) -path P and an (s_2, t_2) -path Q with a set of specified vertices $X \subseteq V(P) \cup V(Q)$ such that the vertices of X are from distinct partite sets can be **quasi-merged** into a single $(\{s_1, s_2\}, \{t_1, t_2\})$ -path R covering X . Such a quasi-merging can be performed in $O(|V(P) \cup V(Q)|)$ time.*

A simple but powerful idea that exploits the high level of adjacency in SMDs and allows us to merge two paths into one without losing anything "important". Since we needed polynomial runtimes for most of our results it turned out to be more convenient to reprove some known results for SMDs or MTs by repeated application of quasi-merging, than analysing the original proofs runtime would have been.

Our main contribution is a polynomial algorithm.

Theorem 5 *There exists an $O(|V|^{28})$ algorithm that decides if a semicomplete multipartite digraph contains an $[x, y]$ -quasi-hamiltonian path and finds one if one exists.*

This is obviously not best possible in matters of runtime. A better algorithm, based on a characterization of all digraphs that do not contain an $[x, y]$ -quasi-hamiltonian path, is available to us, but the characterization and its proof are heavily case-based and contain long passages of technical arguments that do not grant interesting insights.

From our algorithm for the $[x, y]$ -quasi-hamiltonian path problem, it is not difficult to derive a polynomial algorithm that finds an $[x, y]$ -path maximizing the number of partite sets intersected.

Finding the longest $[x, y]$ -path in a semicomplete digraph can be done in polynomial time. Generalizing this to SMDs seems promising.

Conjecture 6 *There exists a polynomial algorithm for finding the longest $[x, y]$ -path in a semicomplete multipartite digraph.*

One could also consider a condition between quasi-hamiltonian and hamiltonian and demand a certain number of vertices from each partite set.

Problem 7 *Is there a polynomial algorithm that given an integer k and a semicomplete multipartite digraph D with vertices x, y and partite sets V_1, \dots, V_c , decides whether D has an $[x, y]$ -path covering at least $\min\{k, |V_i|\}$ vertices from each V_i , $i = 1, \dots, c$?*

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Balanced valuations and circular flows

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EXTENDED ABSTRACT

We consider circulations on finite (multi-) graphs $G = (V, E)$ with vertex set V and edge set E . An *orientation* D of G is an assignment of a direction to each edge, and for $v \in V$, $D^-(v)$ ($D^+(v)$) is the set of edges whose head (tail) is v . The oriented graph is denoted by $D(G)$, $d_{D(G)}^-(v) = |D^-(v)|$ and $d_{D(G)}^+(v) = |D^+(v)|$ denote the *indegree* and *outdegree* of vertex v in $D(G)$, respectively. The degree of a vertex v of the undirected graph G is denoted by $d_G(v) = d_{D(G)}^+(v) + d_{D(G)}^-(v)$.

Let φ be a function from the edge set E of the oriented graph $D(G) = (V, E)$ into the real numbers; (D, φ) is a *nowhere-zero r -flow* on $G = (V, E)$, if $1 \leq \varphi(e) \leq r - 1$, for all $e \in E$, and $\sum_{e \in D^+(v)} \varphi(e) = \sum_{e \in D^-(v)} \varphi(e)$, for all $v \in V$.

In 1975, Jaeger [1] introduced the concept of a *balanced valuation* of a graph G . A *balanced valuation* on G is a function w from the vertex set $V(G)$ into the real numbers, such that for all $X \subseteq V(G)$: $|\sum_{v \in X} w(v)| \leq |\partial_G(X)|$, where $\partial_G(X)$ is the set of edges with precisely one end in X . The following fundamental theorem of Jaeger shows that circular flows and balanced valuations are equivalent concepts.

Theorem 1 ([1]) *Let $r > 2$ be a rational number. A multigraph $G = (V, E)$ has a nowhere-zero r -flow if and only if there is a balanced valuation w of G , such that for each $v \in V$, there is an integer $k_v \equiv d_G(v) \pmod{2}$ and $w(v) = k_v \frac{r}{r-2}$.*

There are some long standing open conjectures on nowhere-zero flows on graphs, e.g.

Conjecture 2 ([4]) *If G is a bridgeless graph, then G has a nowhere-zero 5-flow.*

Conjecture 3 ([5]) *If G is a bridgeless graph without 3-edge cuts, then G has a nowhere-zero 3-flow.*

These conjectures were originally formulated as integer flow problems. Seymour[3] proved, that every bridgeless graph has a nowhere-zero 6-flow. In the recent years nowhere-zero flows with rational flow values have been studied and balanced valuations are a quite powerful tool to prove results. We will give an overview of results on nowhere-zero flows on graphs, which are proved with the help of balanced valuations.

We first determine the circular flow number $F_c(G)$ (which is the $\inf\{r \mid (D, \varphi) \text{ is a nowhere-zero } r\text{-flow on } G\}$) of some classes of graphs. Let \mathcal{F}^c be the set of circular flow numbers of graphs and \mathcal{F}_d^c be the set of circular flow number of d -regular graphs.

Theorem 4 *For all $k \geq 1$: $\mathcal{F}_{2k}^c = \{2\}$ and $\mathcal{F}_{2k+1}^c = (\mathcal{F}^c - [2; 2 + \frac{2}{2k-1}]) \cup \{2 + \frac{1}{k}\}$.*

The oddness $\omega(G)$ of a bridgeless cubic graph is the minimum number of odd circuits in a 2-factor of G . We prove

Theorem 5 *Let G be a cyclically k -edge connected graph. If $k \geq \frac{5}{2}\omega(G) - 3$, then $F_c(G) \leq 5$, and if $k \geq \frac{3}{2}\omega(G) + 1$, then $F_c(G) < 6$.*

We generalize Jaeger's concept [2] of nearly nowhere-zero k -flow graphs to circular flows and show for k -edge connected graphs G ($k \geq 2$): (1) If G has an edge e such that $F_c(G - e) \leq (1 - 1/k)r$, then $F_c(G) \leq r$. (2) If $r \geq 3$ is an integer and G has an edge e such that $F_c(G) - e < r$, then $F_c(G) \leq r$.

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FPT algorithms for domination in biclique-free graphs

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EXTENDED ABSTRACT

A class of graphs is said to be biclique-free if there is an integer t such that no graph in the class contains $K_{t,t}$ as a subgraph. Large families of graph classes, such as any nowhere dense class of graphs or d -degenerate graphs, are biclique-free. We show that various domination problems are fixed-parameter tractable on biclique-free classes of graphs, when parameterizing by both solution size and t . In particular, the problems k -DOMINATING SET, CONNECTED k -DOMINATING SET, INDEPENDENT k -DOMINATING SET and MINIMUM WEIGHT k -DOMINATING SET are shown to be FPT, when parameterized by $t + k$, on graphs not containing $K_{t,t}$ as a subgraph. With the exception of CONNECTED k -DOMINATING SET all described algorithms are trivially linear in the size of the input graph.

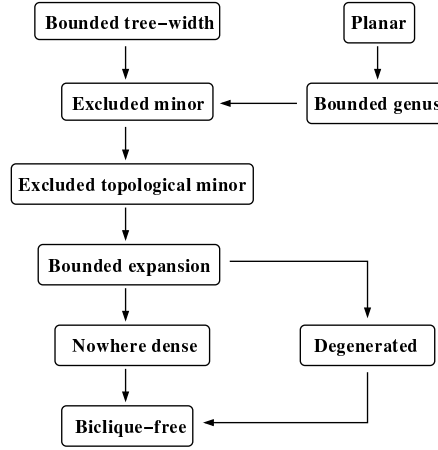


Figure 15: Inclusion relations between some of the mentioned graph class properties. We refer to Nesetril and Ossona de Mendez [10] for a more refined view.

The k -dominating set problem is notorious in the theory of fixed-parameter tractability (see [6, 11, 8] for an introduction to parameterized complexity). It was the first problem to be shown $W[2]$ -complete [6], and it is hence unlikely to be FPT, i.e. unlikely to have an algorithm with runtime $f(k)n^c$ for f a computable function, c a constant and n the number of vertices of the input graph. However, by restricting the class of input graphs, say to planar graphs, we can obtain FPT algorithms [1], even if the problem remains NP-complete on planar graphs [9]. The parameterized complexity of the dominating set problem has been heavily studied with the tractability frontier steadily pushed forward by enlarging the class of graphs under consideration. One such line of improvements for k -dominating set consists of the series of FPT algorithms starting with planar graphs by Alber et al. [1], followed by bounded genus graphs by Ellis et al. [7], H -minor free graphs by Demaine et al. [5], bounded expansion graphs by Nesetril and Ossona de Mendez [10], and culminating in the FPT algorithm for nowhere dense classes of graphs by Dawar and Kreutzer [4]. See Figure . Alon and Gutner [2] have shown that k -dominating set on d -degenerate graphs parameterized by $k + d$ is FPT. A relation between the different graph class properties can be seen in Figure .

In this paper, we push the tractability frontier forward by considering the above variants of k -dominating set on t -biclique free graphs and showing that they are FPT when parameterized by $k + t$. Our algorithms are simple and rely on results from extremal graph theory

that bound the number of edges in a t -biclique free graph. Bollobás in his book “Extremal Graph Theory” discusses the so called Zarankiewicz Problem of giving an upper bound for the number of edges in graphs where $K_{s,t}$ is forbidden as a subgraph for integers s, t , and the following proposition, which he attributes to Kövari, Sós and Turán, turns out to be very useful when studying graphs without $K_{s,t}$ as a subgraph.

Proposition 1 (Bollobás VI.2) *For integers s, t let $G = (V_1, V_2, E)$ be a bipartite graph not containing $K_{s,t}$ as a subgraph where $|V_1| = n_1$ and $|V_2| = n_2$. Then for $2 \leq s \leq n_2$ and $2 \leq t \leq n_1$ we have that $|E| < (s-1)^{\frac{1}{t}}(n_2-t+1)n_1^{1-\frac{1}{t}} + (t-1)n_1$.*

A famous open problem in parameterized complexity is the question if there is an FPT algorithm deciding if a graph G is k -biclique-free and we briefly mention this open problem in light of our algorithms.

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